Division Algebras: Family Replication

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The link of the Division Algebras to 10-dimensional spacetime and one leptoquark family is extended to encompass three leptoquark families.

Introduction

No volume devoted to noncommutative mathematical structures of any sort would be complete without some attention paid to the quaternions ($\mathbf{H}$) and octonions ($\mathbf{O}$). These, together with the commutative algebras $\mathbf{R}$ (real numbers) and $\mathbf{C}$ (complex numbers), are the four real normed division algebras, linked to or the source of much that is elegant, noteworthy and exceptional in mathematics. So generative are they that many have been led to believe that these algebras are, or ought to be, as inextricably intertwined in the laws of physics as they are in mathematics, and certainly this is true of $\mathbf{R}$ and $\mathbf{C}$. I shall present here some ideas in support of the broader notion, ideas requiring all of these algebras, including the noncommutative quaternions, and the noncommutative and nonassociative octonions. References will be scarce here, and I direct the reader to John Baez's excellent review for a broader discussion of quaternions and octonions $[\mathbf{1}]$.

This article is an extension of work begun in $[\mathbf{2}]$, which builds on $[\mathbf{3}]$. There it is assumed that none of the three hypercomplex division algebras ($\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$) is any more singular than any other, and all have a role to play in the design of our physical reality. Together with their obvious links to the groups of the leptoquark Standard Model, I found it natural to consider

$$\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$$

as a starting point for a physical theory ($\mathbf{T}$ is nothing more than the complexification of the quaternionization of the octonions).

Each of these algebras, $\mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$, and the tensor product $\mathbf{T}$, is a spinor space (see $[\mathbf{2}]$). A Dirac algebra spinor of a fundamental fermion like the electron is conventionally represented as an element of $\mathbf{C^4}$, and so is 8 dimensional over the reals. $\mathbf{T}$ is 64-dimensional, and $\mathbf{T^2}$, a complexified $\mathbf{R^{1,9}}$ spinor, is 128-dimensional. Well, $128 = (8+8) \times 8$, and associated with the first family of leptons and quarks there are 8 fermions and 8 antifermions (neutrinos treated as Dirac throughout). As has been

shown in [**2**] and [**3**] and elsewhere, there is more to this than the numerological coincidence that $\mathbf{T^2}$ has the dimensionality required to account for 16 Dirac spinors. There are few if any of the mathematical features of the Standard Model of quarks and leptons that do not fall naturally out of a theory based mathematically on the spinor space $\mathbf{T^2}$. For me the correspondence was like the bite of a salt water crocodile: powerless to resist, I surrendered to its grasp and devoted several years to exploring the relationship. During that time there were two things that periodically bothered me:
• Why $\mathbf{T^2}$, and not just $\mathbf{T}$?
• And where were families two and three?

Mathematical Motivation

Let PSO_k be a Projective Special Orthogonal group. In [**4**] it is noted that:
•**1** the existence of **C** implies PSO_2 is commutative;
•**2** the existence of **H** implies $PSO_4 \cong PSO_3 \times PSO_3$;
•**3** the existence of **O** implies PSO_8 has a triality automorphism of order 3.

•**1** The Special Orthogonal group SO_2 acting on 2^1-dimensional **C** can be represented by a single complex unit, $u + iv$, where $u^2 + v^2 = 1$.
•**2** Any element of the group SO_4 acting on the 2^2-dimensional **H** requires at most 16 quaternion doublets, (u, v), summing actions of the form

$$x \longrightarrow u_L v_R[x] = uxv.$$

This becomes somewhat obvious when we realize that SO_4 is generated by the six elements q_{Li} and q_{Ri}, where q_i, i = 1,2,3, are the three imaginary quaternion units, and q_{Li} and q_{Ri} are the maps

$$X \longrightarrow q_i X, \qquad X \longrightarrow X q_i,$$

for X in **H**. A general element of SO_4 takes the form $u + v^i q_{Li} + r^j q_{Rj} + s^{ij} q_{Li} q_{Rj}$ (1+3+3+9 = 16), with suitable conditions placed on the coefficients.
•**3** Any element of the group SO_8 acting on the 2^3-dimensional **O** requires at most 64 octonion triples, (u, v, w), summing actions of the form

$$X \longrightarrow u_L v_L w_L[X] = u(v(wX)).$$

This becomes somewhat obvious when we realize that SO_8 is generated by the 28 elements e_{La} and e_{Lab}, where e_a, a = 1,...,7, are the imaginary octonion units, and e_{La} and e_{Lab} (and e_{Labc}) are the maps

$$X \longrightarrow e_a X, \qquad X \longrightarrow e_a(e_b X), \qquad X \longrightarrow e_a(e_b(e_c X)),$$

for X in **O** (no right multiplication is needed: see [**2**]). A general element of SO_8 takes the form $u + v^a e_{La} + r^{ab} e_{Lab} + s^{abc} e_{Labc}$ (1 + 7 + 21 + 35 = 64), with suitable conditions placed on the coefficients.

On page 12 of Conway and Sloane's *Sphere Packings...* (**[5]**) three laminated lattices are singled out for their density and simplicity. These are:
•**1** $\Lambda_2 = A_2$, which can be represented in $\mathbf{C^1}$;
•**2** $\Lambda_8 = E_8$, which can be represented in $\mathbf{H^2}$ (and $\mathbf{O^1}$);
•**3** Λ_{24} (Leech lattice), which can be represented in $\mathbf{O^3}$.
In each case the representations are intimately related to certain integral elements over the underlying division algebras (**[4,5]**).

3 Family Model

This list could go on, but the point it is attempting to make is that mathematically $\mathbf{C^1}$, $\mathbf{H^2}$ and $\mathbf{O^3}$ are each in a way special, associated with exceptional mathematical structures. Once I let this idea soak in, it resolved the two outstanding issues mentioned above:
Why $\mathbf{T^2}$, and not just $\mathbf{T}$? Because it's not $\mathbf{T^2}$ at all, but rather

$$\mathbf{T^6} = \mathbf{C^1} \otimes \mathbf{H^2} \otimes \mathbf{O^3}.$$

And where were families two and three? Well, if $\mathbf{T^2}$ accounts for one full family and its antifamily, then $\mathbf{T^6}$ would account for three.

Some notation:

- $\mathbf{K}_L, \mathbf{K}_R$ - the algebras of left and right actions of an algebra **K** on itself.
- $\mathbf{K}_A$ - the algebra of combined left and right actions of an algebra **K** on itself.
- **K**(m) - m$\times$m matrices over the algebra **K**;
- $\mathbf{K}^m$ - and m$\times$1 column over **K**;
- $\mathcal{CL}$(p,q) - the Clifford algebra of the real spacetime with signature (p+,q-).

If we let $\mathbf{P} = \mathbf{C} \otimes \mathbf{H}$, then $\mathbf{P}_L$ is isomorphic to the Pauli algebra, so $\mathbf{P}_L(2)$ is isomorphic to the Dirac algebra, and $\mathbf{H}_R$, which commutes with $\mathbf{P}_L(2)$ (which acts on $\mathbf{H}^2$), provides an internal $SU(2)$ degree of freedom.

One can do much the same thing **[2,3]** with $\mathbf{T}$. $\mathbf{T}_L$ is a Pauli-like algebra, and $\mathbf{T}_L(2)$ is the Dirac algebra of 1,9-spacetime. Again there remains the internal $\mathbf{H}_R$ commuting with $\mathbf{T}_L(2)$, providing an isospin $SU(2)$. The associated spinor space ($\mathbf{T}^2$) transforms with respect to the standard symmetry as the direct sum of a leptoquark family and antifamily of 1,3-Dirac spinors.

Obviously, since $\mathbf{T}^2$ accounts for one family/antifamily, $\mathbf{T}^6$ would account for three, which is the accepted number of total families. However, in **[2]** the algebra $\mathbf{T}_L$(2), which acts on $\mathbf{T}^2$, is isomorphic to a Clifford algebra (the complexification of $\mathcal{CL}$(1,9)). Since all Clifford algebras are 2^k-dimensional, the $3^2 2^{13}$-dimensional $\mathbf{T}_A$(6) (which is the full algebra of actions associated with $\mathbf{T}^6$) is not a Clifford algebra.

Let's plow ahead anyway, and first look at the 2^{15}-dimensional $\mathbf{T}_A(4)$, isomorphic to the complexification of $\mathcal{CL}$(1,13). This acts on $\mathbf{T}^4$, which is a pair of leptoquark families (and their antifamilies).

Some 2×2 real matrices:

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Define, for example, the following 4×4 real matrix:

$$[\beta \otimes \alpha] = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}.$$

We'll use the $\mathcal{CL}$(1,13) 1-vector basis (let $\mathcal{CL}_k$(p,q) be the k-vector basis of $\mathcal{CL}$(p,q)):

$$[\epsilon \otimes \beta](iq_{R3}), \;\; [\epsilon \otimes \gamma]q_{Lk}e_{L7}(iq_{R3}), \; k = 1,2,3, \;\; [\epsilon \otimes \gamma]ie_{Lp}(iq_{R3}), \; p = 1,...,6,$$

$$[\beta \otimes \epsilon]q_{R1}, \; [\beta \otimes \epsilon]q_{R2}, \; [\beta \otimes \alpha]q_{R3}, \; [\gamma \otimes \alpha].$$

The first line contains 10 elements which generate a $\mathcal{CL}$(1,9) subalgebra of $\mathcal{CL}$(1,13). This is essentially the $\mathcal{CL}$(1,9) that appeared in **[2]**. The second line contains 4 elements which generate a $\mathcal{CL}$(0,4) subalgebra. Under the commutator product the associated 2-vectors are a basis for $so(4) \sim su(2) \times su(2)$. The six generators are:

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\{[\epsilon \otimes \alpha]q_{R1}, \; [\epsilon \otimes \alpha]q_{R2}, \; [\epsilon \otimes \epsilon]q_{R3}\}.$$

The 4×4 real matrix $[\alpha \otimes \epsilon]$ is the product of the last four 1-vectors above, hence it commutes with the $\mathcal{CL}$(1,9) 1-vectors, but anticommutes with the $\mathcal{CL}$(0,4) 1-vectors. Therefore it can be used to reduce the 1,13-spacetime to 1,9-spacetime. In particular, at the 1-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\mathcal{CL}_1(1,13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = \mathcal{CL}_1(1,9)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]).$$

At the 2-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])so(1,13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = (so(1,9) \times su(2))\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]),$$

each projector $\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])$ picking out an $su(2)$ half of $so(4)$, and projecting from the spinor space, $\mathbf{T}^4$, a $\mathbf{T}^2$ subspace. Hence this reduction results in exactly the scenario developed in **[2]**, except doubled. Projection operators, $\rho_{L\pm} = \frac{1}{2}(1 \pm ie_{L7})$ and $\rho_{R\pm} = \frac{1}{2}(1 \pm ie_{R7})$, further reduce the $\mathcal{CL}_1$(1,9) to $\mathcal{CL}_1$(1,3), and yielding a total Lie algebra reduction:

$$so(1,13) \longrightarrow so(1,9) \times su(2) \longrightarrow so(1,3) \times u(1) \times su(2) \times su(3).$$

The associated $\mathbf{T}^2$ subspace is the direct sum of a family and antifamily of leptons and quarks, transforming appropriately with respect to $u(1) \times su(2) \times su(3)$.

With a Clifford algebra and spinors we can form a Dirac operator and Lagrangian. If there were 2^k families then $\mathbf{T}^{2k}$ would be the appropriate hyperspinor space, acted on by a conventional Clifford algebra. But it is believed there are exactly 3 families, and we will have to get a little creative in constructing a Dirac-like Lagrangian for this case.

A Dirac operator for the $\mathcal{CL}$(1,13) 2-family model developed above would be

$$\begin{bmatrix} \partial\!\!\!/_{1,9} & \partial\!\!\!/^{+}_{0,4} \\ \partial\!\!\!/^{-}_{0,4} & \partial\!\!\!/_{1,9} \end{bmatrix},$$

built from the original set of 14 1-vectors. (As noted in **[6]**, this leads to interfamily mixing, including neutrinos.) For the 3-family case, one suggestion is to incorporate 3 of these 2-family Dirac operators into something new, motivated by the form of matrices in the exceptional Jordan algebra. In particular, consider a Lagrangian term like

$$\overline{\Psi}\mathcal{D}\Psi = \begin{bmatrix} \overline{\psi_1} & \overline{\psi_2} & \overline{\psi_3} \end{bmatrix} \begin{bmatrix} \partial\!\!\!/_{1,9} & \partial\!\!\!/^{+}_{0,4} & \partial\!\!\!/^{-}_{0,4} \\ \partial\!\!\!/^{-}_{0,4} & \partial\!\!\!/_{1,9} & \partial\!\!\!/^{+}_{0,4} \\ \partial\!\!\!/^{+}_{0,4} & \partial\!\!\!/^{-}_{0,4} & \partial\!\!\!/_{1,9} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$
$$= \overline{\psi_1}\ \partial\!\!\!/_{1,9}\psi_1 + \overline{\psi_1}\ \partial\!\!\!/^{+}_{0,4}\psi_2 + \ldots$$

Each of the $\psi_k, k = 1,2,3$, is a complete leptoquark family plus antifamily residing in a copy of $\mathbf{T^2}$. There are three terms like $\overline{\psi_1}\ \partial\!\!\!/_{1,9}\psi_1$, from which we derive single family interactions (**[2,3]**), and six of the form $\overline{\psi_1}\ \partial\!\!\!/^{+}_{0,4}\psi_2$, which mixes two different families - on the assumption $\partial\!\!\!/^{+}_{0,4}\psi_2 \neq 0$ (see **[6]**).

If this approach is valid there is much that needs to be done to complete the picture. In particular, there is no single conventional pseudo-orthogonal space associated with the operator $\mathcal{D}$. Should the three $\partial\!\!\!/_{1,9}$ on the diagonal, and three $\partial\!\!\!/^{+}_{0,4}$ off-diagonal, be distinct? How does one obtain bivectors leading to Lie group actions and internal symmetries?

There are other 3×3 structures that may be relevant in this context, including a kind of Fermionic Clifford algebra related to supersymmetry **[3]**. This has not been pursued to this point. The crocodiles grip, while still strong, has perforce been largely ignored for some little time.

Appendix

$$so_2$$

$$i.$$

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$$so_8$$

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \{q_{Li}, q_{Rj}\},$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \{1, q_{Li}q_{Rj}\}.$$

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$$so_{24}$$

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \{e_{La}, e_{Lab}\},$$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \{e_{La}, e_{Lab}\},$$

$$\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \{1, e_{Labc}\}.$$

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References:

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[3] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics,* (Kluwer, 1994).

[4] J.H. Conway, D.A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry,* (A K Peters, 2003).

[5] J.H. Conway, N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, (Springer-Verlag, 1993).

[6] G.M. Dixon, www.7stones.com/Homepage/14mix.pdf