Division Algebras, Clifford Algebras, Periodicity

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$\prec$ Last modified: 2012.04.04 $\succ$

Thoughts on periodicities of order 2^k and 24.

Euclidean Clifford algebras

Notation:

$$\mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}$$

real numbers, complex numbers, quaternions and octonions.

$$\mathbf{K}_L,\ \mathbf{K}_R,\ \mathbf{K}_A$$

the algebras of left-sided, right-sided, and both-sided actions of a division algebra $\mathbf{K}$ on itself.

$$\mathcal{CL}(p, q)$$

the Clifford algebra of a p,q-pseudo-orthogonal space with metric signature, p(+), q(-).

$$\mathbf{K}(n)$$

the algebra of $n \times n$ matrices over the division algebra $\mathbf{K}$.

$$^2\mathbf{K}(n)$$

the block diagonal $2n \times 2n$ matrices over $\mathbf{K}(n)$: (so $2n^2$-dimensional).

In particular, given this basis for $\mathbf{R}$(2),

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

we have this as a basis for $^2\mathbf{R}$,

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

My introduction to both division algebras and Clifford algebras was [1]. The following list of Clifford algebra isomorphisms derives from that source:

k	$\mathcal{CL}(0,k)$		$\mathcal{CL}(k,0)$
0		$\mathbf{R}$	
1	$\mathbf{C}$		${}^2\mathbf{R}$
2	$\mathbf{H}$		$\mathbf{R}(2)$
3	${}^2\mathbf{H}$		$\mathbf{C}(2)$
4		$\mathbf{H}(2)$	
5	$\mathbf{C}(4)$		${}^2\mathbf{H}(2)$
6	$\mathbf{R}(8)$		$\mathbf{H}(4)$
7	${}^2\mathbf{R}(8)$		$\mathbf{C}(8)$
8		$\mathbf{R}(16)$	

However, we can dispense with all matrix algebras by making use of *split* versions of the division algebras. Bases for **C**, **H** and **O** are

$$\begin{aligned}&\mathbf{C}:\ \{1,\ i\};\\ &\mathbf{H}:\ \{q_0=1,\ \ q_1,\ \ q_2,\ \ q_3\};\\ &\mathbf{O}:\ \{e_0:=1,\ \ e_1,\ \ e_2,\ \ e_3,\ \ e_4,\ \ e_5,\ \ e_6,\ \ e_7\}\end{aligned}$$

(see [2] and [3] for multiplication tables and much much more). We now need a new copy of the complex algebra, and we'll denote its imaginary unit ι (so $\iota^2=-1$, and ι commutes with everything, but it is not the same as our original complex unit i). Then bases for split versions of those division algebras (using the multiplication tables in [2] and [3]) are

$$\begin{aligned}&\tilde{\mathbf{C}}:\ \{1,\ \iota i\};\\ &\tilde{\mathbf{H}}:\ \{q_0=1,\ \ q_1,\ \ \iota q_2,\ \ \iota q_3\};\\ &\tilde{\mathbf{O}}:\ \{e_0:=1,\ \ e_1,\ \ e_2,\ \ \iota e_3,\ \ e_4,\ \ \iota e_5,\ \ \iota e_6,\ \ \iota e_7\}\end{aligned}$$

(although these are in fact real algebras, they are no longer division algebras).

We can dispense with matrix algebras making use of the following isomorphisms and equivalencies:

$$\begin{gathered}\tilde{\mathbf{C}}\simeq{}^2\mathbf{R}\\ \mathbf{H}\simeq\mathbf{H_L}\simeq\mathbf{H_R}\\ \tilde{\mathbf{H}}\simeq\tilde{\mathbf{H}}_L\simeq\tilde{\mathbf{H}}_R\simeq\mathbf{R}(2)\\ \tilde{\mathbf{H}}^2\simeq\mathbf{H}^2\simeq\mathbf{H}_A\simeq\mathbf{R}(4)\\ \mathbf{O}_L=\mathbf{O}_R=\mathbf{O}_A\simeq\mathbf{R}(8)\end{gathered}$$

In this, and what follows, it is understood that $\mathbf{K}^n:=\mathbf{K}\otimes\mathbf{K}\otimes\ldots\otimes\mathbf{K}$, where there are n distinct copies of $\mathbf{K}$ on the righthand side (see [2] and [3]).

We can now write that Clifford algebra representation table without matrices:

k	$\mathcal{CL}(0,k)$		$\mathcal{CL}(k,0)$
0		$\mathbf{R}$	
1	$\mathbf{C}$		$\tilde{\mathbf{C}}$
2	$\mathbf{H}$		$\tilde{\mathbf{H}}$
3	$\tilde{\mathbf{C}} \otimes \mathbf{H}$		$\tilde{\mathbf{H}} \otimes \mathbf{C}$
4		$\mathbf{H} \otimes \tilde{\mathbf{H}}$	
5	$\mathbf{C} \otimes \mathbf{H} \otimes \mathbf{H}$		$\mathbf{H} \otimes \tilde{\mathbf{H}} \otimes \tilde{\mathbf{C}}$
6	$\mathbf{O_L}$		$\mathbf{H}^3$
7	$\tilde{\mathbf{C}} \otimes \mathbf{O_L}$		$\mathbf{O_L} \otimes \mathbf{C}$
8		$\mathbf{H}^4 \simeq \mathbf{H}_A^2 \simeq \mathbf{O_L} \otimes \tilde{\mathbf{H}}$	

To better illustrate what's going on here, I'll rewrite the above a little more schematically, using some different isomorphisms:

				$\mathcal{CL}(0,k)$	k	$\mathcal{CL}(k,0)$				
				$\mathbf{R}$	0	$\mathbf{R}$				
			$\mathbf{C}$	$\mathbf{R}$	1	$\mathbf{R}$	$\tilde{\mathbf{C}}$			
			$\mathbf{H_L}$	$\mathbf{R}$	2	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H_L}$	$\mathbf{R}$	3	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	4	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	5	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{C}}$	
	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	6	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\mathbf{R}$	7	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H_L}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_A^2$	8	$\mathbf{H}_A^2$				

There are some striking periodicities in these two tables. Modulo 2 we see that going from $k = 2n$ to $k = 2n + 1$ we alternately add $\mathbf{C}$ or $\tilde{\mathbf{C}}$, which depending on if we are looking at the $\mathcal{CL}(0,k)$ column, or $\mathcal{CL}(k,0)$. Modulo 4 we see that

$$\mathcal{CL}(0,4n) \simeq \mathcal{CL}(4n,0),\ n \geq 0.$$

Modulo 8 is the big periodicity, related to what is known as Bott periodicity. In this context we first see that at $k = 8$ there is a kind of algebraic collapse, or simplification, in the representation. But also,

$$\mathcal{CL}(0,k+8) \simeq \mathcal{CL}(0,k) \otimes \mathcal{CL}(0,8),$$
$$\mathcal{CL}(k+8,0) \simeq \mathcal{CL}(k,0) \otimes \mathcal{CL}(0,8).$$

This kind of order 8 periodicity applies as well to $\mathcal{CL}(p,q)$, with neither p nor q equal to 0, but I'm not interested in that here.

Let's take a look at the second of the tables above and expand it all the way to $k = 24$:

				$\mathcal{CL}(0,k)$	k	$\mathcal{CL}(k,0)$				
				$\mathbf{R}$	0	$\mathbf{R}$				
			$\mathbf{C}$	$\mathbf{R}$	1	$\mathbf{R}$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	2	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	3	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	4	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	5	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_{\mathbf{L}}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	6	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_{\mathbf{L}}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\mathbf{R}$	7	$\mathbf{R}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_{\mathbf{L}}$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_A^2$	8	$\mathbf{H}_A^2$				
			$\mathbf{C}$	$\mathbf{H}_A^2$	9	$\mathbf{H}_A^2$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_L$	$\mathbf{H}_A^2$	10	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\mathbf{H}_A^2$	11	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^2$	12	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^2$	13	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^2$	14	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^2$	15	$\mathbf{H}_A^2$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{H}_A^4$	16	$\mathbf{H}_A^4$				
			$\mathbf{C}$	$\mathbf{H}_A^4$	17	$\mathbf{H}_A^4$	$\tilde{\mathbf{C}}$			
			$\mathbf{H}_L$	$\mathbf{H}_A^4$	18	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$			
		$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\mathbf{H}_A^4$	19	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$		
		$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^4$	20	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$		
	$\mathbf{C}$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^4$	21	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{C}}$	
	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^4$	22	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	
$\tilde{\mathbf{C}}$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\mathbf{H}_A^4$	23	$\mathbf{H}_A^4$	$\tilde{\mathbf{H}}_L$	$\mathbf{H}_L$	$\tilde{\mathbf{H}}_L$	$\mathbf{C}$
				$\mathbf{O}_L^4$	24	$\mathbf{O}_L^4$				

This table makes the order 8 periodicity very pronounced. At every multiple of 8 there is a kind of algebraic collapse/simplification, after which we start adding things in the same way as we did previously. Keep in mind that few of these representations are unique. For example, at k = 16,

$$\mathbf{H}_A^4 \simeq \mathbf{H}_L^8 \simeq \mathbf{H}_A \otimes \mathbf{O}_L^2.$$

So the octonion algebra could have been introduced before k = 24.

The question is: is this order 24 algebraic collapse to a product of just octonions (left actions) meaningful? Well, ... Let's take a look at a 1-vector basis for the Clifford algebra $\mathcal{CL}(24,0)$ represented by $\mathbf{O}_L^4$. We need four copies of $\mathbf{O}$, and we'll denote their bases by

$$^{m}e_a,\ a = 0, ..., 7,\ \ m = 1, 2, 3, 4.$$

This is the $\mathcal{CL}(24,0)$ 1-vector basis I came up with ($p = 1, ..., 6$):

$$\begin{array}{cccc} {}^1e_{Lp} & {}^2e_{L7} & {}^3e_{L0} & {}^4e_{L0} \\ {}^1e_{L0} & {}^2e_{Lp} & {}^3e_{L7} & {}^4e_{L0} \\ {}^1e_{L7} & {}^2e_{L0} & {}^3e_{Lp} & {}^4e_{L0} \\ {}^1e_{L7} & {}^2e_{L7} & {}^3e_{L7} & {}^4e_{Lp} \end{array}$$

This gives us 24 anti-commuting elements of $\mathbf{O}_L^4$ (6 for each row). The product of all 24 is

$$\pm{}^1e_{L7}{}^2e_{L7}{}^3e_{L7}{}^4e_{L7}.$$

Interestingly, if we replace $\mathbf{O}$ by $\mathbf{H}$ (that is, $\mathbf{H}_L^4$), and build a similar basis for a Clifford using quaternions instead of octonions ($r = 1, 2$), we get

$$\begin{array}{cccc} {}^1q_{Lr} & {}^2q_{L3} & {}^3q_{L0} & {}^4q_{L0} \\ {}^1q_{L0} & {}^2q_{Lr} & {}^3q_{L3} & {}^4q_{L0} \\ {}^1q_{L3} & {}^2q_{L0} & {}^3q_{Lr} & {}^4q_{L0} \\ {}^1q_{L3} & {}^2q_{L3} & {}^3q_{L3} & {}^4q_{Lr} \end{array}$$

which is a basis for $\mathcal{CL}(8,0)$. So the octonions are associated with $\mathcal{CL}(24,0)$, and the quaternions with $\mathcal{CL}(8,0)$, at least within this context.

[Note added 2014.03.12]
24 is the smallest dimension k for which

$$\mathcal{CL}(k,0) \simeq \mathcal{CL}(0,k),$$

and both can be represented purely in terms of $\mathbf{O}_L$ ($\mathbf{O}_L^4$). The first dimension in which any Clifford algebra can feature the full $\mathbf{O}_L$ in its representation is $n = 6$. And the first dimension in which all $\mathcal{CL}(p,q)$ can exploit $\mathbf{O}_L$ as part of their representations is $p + q = 8$.

$$24 = LCM(6,8).$$

Half the Story

Clifford algebras are part of a broader context which includes its spinor space. When exploited in theoretical physics Clifford algebras are generally represented as real or complex matrices. In the former case the spinor space would usually be a column matrix of real numbers, and in the latter case a column of complex numbers. And there, at least as far as the mathematics is concerned, the matter would rest.

But Clifford algebras can also be represented using the division algebras **H** and **O** [1] [2] [3], and frequently in these cases there is more to the mathematics than just a matrix algebra acting on a simple spinor space, and the reason for this is that the spinor spaces are no longer so simple.

We see this in the simplest case of $\mathcal{CL}(0,2)$ represented over $\mathbf{H}_L$, where a 1-vector basis for this Clifford algebra could be $\{q_{L1},\ q_{L2}\}$, with a 2-vector basis consisting of just q_{L3}, which generates the Lie group $U(1) \simeq SO(2)$.

In this case, however, because multiplication of **H** from the right commutes with all multiplication from the left, we have an entire copy of quaternion algebra action, represented by $\mathbf{H}_R$, which is internal to the Clifford algebra action, represented by $\mathbf{H}_L$. In [2] and [3] this gives rise in theory building to an $SU(2)$ identified with isospin.

As another example, and one quite distinct from that quaternion case, we begin by defining

$$\mathbf{S} := \mathbf{C} \otimes \mathbf{O}.$$

Then the Clifford algebra $\mathcal{CL}(7,0)$ can be represented by $\mathbf{S}_L$, which is also isomorphic to the 2^7 = 128-dimensional complex matrix algebra, **C**(8). An obvious 1-vector basis consists of the 7 elements $ie_{La},\ a = 1,...,7$. In this case the spinor space is the 16-dimensional algebra **S**. In the **C**(8) case the spinor space is also 16-dimensional (over **R**) consisting of 8×1 complex column matrices. But there is a huge difference between these two spinor spaces (see in particular [3]). In the complex case the components of the spinor space are complex numbers, and **C** is a division algebra, so its identity can not be resolved into a nontrivial collection of orthogonal idempotents. In the **S** case the spinors components are elements of **S**, which does have a nontrivial resolution of its identity. This is generally represented by the elements

$$\rho_\pm := \frac{1}{2}(1 \pm ie_7),$$

which satisfy

$$\begin{aligned}
&\rho_\pm \rho_\pm = \rho_\pm, \\
&\rho_\pm \rho_\mp = 0, \\
&\rho_+ + \rho - = 1.
\end{aligned}$$

As has been shown [3], with the aid of these projection operators the $SO(7)$ that is generated from the Lie algebra of $\mathcal{CL}(7,0)$ 2-vectors naturally breaks down to $U(1) \times SU(3)$, and the spinor space decomposes into four $SU(3)$ multiplets:

$$\begin{array}{lcl}
\rho_+ \mathbf{S} \rho_+ & : & SU(3) \text{ singlet;} \\
\rho_+ \mathbf{S} \rho_- & : & SU(3) \text{ triplet;} \\
\rho_- \mathbf{S} \rho_- & : & SU(3) \text{ anti-singlet;} \\
\rho_- \mathbf{S} \rho_+ & : & SU(3) \text{ anti-triplet.}
\end{array}$$

Ok, the point is, although much the same thing could be done representing things with complex matrices, and giving the $S^1 \times S^7$ subspace of the spinor space an algebraic structure, the use of **S** makes this inherent algebraic structure in the spinor space considerably more obvious.

So, What About 24

In this case we'll use $\mathbf{O}_L^4$ to represent $\mathcal{CL}(24,0)$, and our spinor space is then

$$\mathbf{L} := \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O}.$$

To make things potentially a little easier to read, we'll let

$$e_a,\ f_a,\ h_a,\ g_a,\ \ a = 0, 1, ..., 7,$$

be the bases for the 4 respective copies of $\mathbf{O}$. Then the basis for our chosen collection of 24 basis 1-vectors for $\mathcal{CL}(24,0)$ is (indices p and q will be understood to run from 1 to 6)

$$\begin{array}{l} e_{Lp}f_{L7}g_{L0}h_{L0}, \\ e_{L0}f_{Lp}g_{L7}h_{L0}, \\ e_{L7}f_{L0}g_{Lp}h_{L0}, \\ e_{L7}f_{L7}g_{L7}h_{Lp}. \end{array}$$

The set of resulting 2-vectors is (a basis for the Lie group $so(24)$)

$$\begin{array}{lr} e_{Lpq}f_{L0}g_{L0}h_{L0}, & so(6), \\ e_{L0}f_{Lpq}g_{L0}h_{L0}, & so(6), \\ e_{L0}f_{L0}g_{Lpq}h_{L0}, & so(6), \\ e_{L0}f_{L0}g_{L0}h_{Lpq}, & so(6), \\ e_{Lp}f_{Lq7}g_{L7}h_{L0}, & 36-d, \\ e_{L7}f_{Lp}g_{Lq7}h_{L0}, & 36-d, \\ e_{Lq7}f_{L7}g_{Lp}h_{L0}, & 36-d, \\ e_{Lq7}f_{L0}g_{L7}h_{Lp}, & 36-d, \\ e_{L7}f_{Lq7}g_{L0}h_{Lp}, & 36-d, \\ e_{L0}f_{L7}g_{Lq7}h_{Lp}, & 36-d. \end{array}$$

The total dimensionality is $4 \times 15 + 6 \times 36 = 276 = \frac{1}{2}24 \times (24-1)$, which is the dimensionality of $so(24)$.

Finally we need a resolution of the identity of $\mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O} \otimes \mathbf{O}$, which I choose to build from:

$$\begin{array}{l} \eta_\pm := \frac{1}{2}(1 \pm e_7h_7), \\ \mu_\pm := \frac{1}{2}(1 \pm f_7h_7), \\ \nu_\pm := \frac{1}{2}(1 \pm g_7h_7). \end{array}$$

Then the resolution consists of the 8 orthogonal idempotents

$$\rho_{\pm\pm\pm} := \eta_\pm\mu_\pm\nu_\pm,$$

where the signs are understood to be independent.

Decomposition of the Spinor Space

A point made repeatedly in my work, and perhaps most cogently in [3], is that when a Clifford algebra's spinor space is composed of a column of tensored division algebras, then there is likely to be a resolution of the identity of that tensored algebra, and this can be used to decompose the pseudo-orthogonal Lie algebra of Clifford algebra bivectors into subalgebras, and decompose the spinor space itself into bits that are multiplets of the Lie algebra decomposition. For example, in [3] it was shown in some detail how the spinor space $\mathbf{S} := \mathbf{C} \otimes \mathbf{O}$ has a resolution of its identity that decomposed the $so(7)$ Lie algebra of bivectors in $\mathbf{S}_L$ down to $u(1) \times su(3)$, and the spinor space $\mathbf{S}$ itself into the direct sum of an $su(3)$ singlet, antisinglet, triplet, and anti triplet.

The spinor space $\mathbf{L}$ is 2^{12}-dimensional. With respect to our resolution of its identity, it can be decomposed into the following $8 \times 8 = 2^6 = 64$ subsets,

$$\rho_{\pm\pm\pm}\mathbf{L}\rho_{\pm\pm\pm},$$

where the 6 signs are independent. These can be collected into 4 groups: 8 for which the three signs on the left match the three signs on the right; $8 \times 3 = 24$ for which one of the signs on the right differs from the corresponding sign on the left; $8 \times 3 = 24$ for which two of the signs on the right differ from the corresponding signs on the left; and 8 for which all the signs on the right differ from the 3 corresponding signs on the left (for a total of $8 + 24 + 24 + 8 = 64$).

2012.04.02

I'm going to end this for the time being. Just to summarize: the associated $so(24)$ reduces with respect to the chosen resolution of the identity to $su(3) \times su(3) \times su(3)$ (with probably some $u(1)$s thrown in), and the spinor space breaks into a bunch of multiplets that are of the form singlet-singlet-singlet, triplet-singlet-singlet, triplet-triplet-singlet, or triplet-triplet-triplet, where it is understood that in the middle two instances we can permute the positions of the multiplet types, and in all cases, any of the multiplets could be replaced with the corresponding anti-multiplet. My reason for stopping this here is an inability to convince myself that the end result is interesting enough to warrant the effort of slogging through the mathematics. It might be, but I don't see it yet. However, I remain convinced that there are the seeds of a periodicity of order 24 in this mathematics.

2012.04.04

Well, it might be more complicated than that. Why aren't there 4 $su(3)$s? Some sharing or absorption of algebras and/or multiplets is going on. Still, ...

Motivation

Topologically Bott periodicity has to do with homotopy groups and the sequences of classical Lie groups, orthogonal, unitary and symplectic. In this context the primary kinds of periodicities that arise are of order 2, 4 and 8.

In the theory of laminated lattices there are also kinds of periodicities of order 2, 4, 8 and 24. It was this that inspired this look at Clifford algebra periodicity, and in particular that $\mathbf{O}_L^4 \simeq \mathcal{CL}(24, 0) \simeq \mathcal{CL}(0, 24)$.

The question naturally arises (at least in my cranium): is there a topological periodicity of order 24?

References

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[2] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*, (Kluwer (now Springer-Verlag), 1994).

[3] G.M. Dixon, *Division Algebras, Lattices, Physics, Windmill Tilting*, (CreateSpace, 2011).