Why is $\mathbf{T}$ Complex?

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The algebra $\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$, which provides an exact fit for the algebraic architecture of the Standard Model of quarks and leptons and their interactions, is a (real) tensor product of all the division algebras. Yet in a mathematical sense it is a complex algebra. Any quantum theory built over $\mathbf{T}$ should therefore be complex, and not quaternion or octonion (it would have to be over some division algebra).

Introduction

The complex algebra, $\mathbf{C}$, is the foundation for analytic function theory and quantum mechanics. Although many have tried, the quaternion algebra, $\mathbf{H}$, in being noncommutative, and the octonion algebra, $\mathbf{O}$, in being both noncommutative and nonassociative, are not fit to play the role $\mathbf{C}$ plays in mathematics and physics. Yet all three are normed division algebras, and quantum theories have been built from all three. It seems, however, that quantum mechanics in our universe is inherently complex.

Each of these division algebras is also a spinor space with respect to the algebra of actions of each on itself. I denote these algebras, $\mathbf{C_A}$, $\mathbf{H_A}$, $\mathbf{O_A}$.

$\mathbf{C_A}$

Because $\mathbf{C}$ is both commutative and associative,

$$\mathbf{C_A} \simeq \mathbf{C},$$

and it makes little sense to differentiate the two. However, $\mathbf{C}$ is 2-dimensional over the reals, $\mathbf{R}$, so should accept a full $\mathbf{R}(\mathbf{2})$ algebra of actions (2×2 matrices over $\mathbf{R}$). This can only be done by augmenting $\mathbf{C_A}$ with complex conjugation. So in this case $\mathbf{C_A}$ is incomplete: it contains no projection operators that can project from all elements of $\mathbf{C}$ their real parts. $\mathbf{C}$ is the end of the line.

$\mathbf{H_A}$

Because $\mathbf{H}$ is not commutative, $\mathbf{H_A}$ is algebraically more interesting and less trivial. In being associative the algebras $\mathbf{H_L}$ and $\mathbf{H_R}$ of left and right actions of $\mathbf{H}$ on itself satisfy

$$\mathbf{H_L} \simeq \mathbf{H_R} \simeq \mathbf{H}.$$

Since $\mathbf{H}$ is a division algebra, it contains no projection operators other than the identity, so $\mathbf{H}$ is what I call left and right incomplete, since the 4-dimensional $\mathbf{H}$ can accept a 16-dimensional set of actions isomorphic to $\mathbf{R}(\mathbf{4})$.

But $\mathbf{H_L}$ and $\mathbf{H_R}$ are distinct, so $\mathbf{H_A}$, which contains both, is bigger than both, and is in fact isomorphic to $\mathbf{R}(\mathbf{4})$. We can therefore decompose the identity of $\mathbf{H_A}$ into 4 orthogonal primitive idempotents, one of which can project from any element of $\mathbf{H}$ its real part (and by virtue of that we can also construct an element of $\mathbf{H_A}$ which when operated on any element of $\mathbf{H}$ will result in its quaternionic conjugate (an anti-automorphism)).

$\mathbf{O_A}$

Like $\mathbf{H}$, $\mathbf{O}$ is noncommutative, so $\mathbf{O_A}$ is a bigger algebra than $\mathbf{O}$ itself. But because $\mathbf{O}$ is nonassociative, $\mathbf{O_L}$ and $\mathbf{O_R}$ are also bigger than $\mathbf{O}$. In fact,

$$\mathbf{O_L} \simeq \mathbf{O_R} \simeq \mathbf{O_A} \simeq \mathbf{R}(\mathbf{8}).$$

So $\mathbf{O}$ is left, right, and in general, complete. But in this case $\mathbf{O_L}$ and $\mathbf{O_R}$ are not distinct: they are the same algebra, and both the same as $\mathbf{O_A}$.

The identity of $\mathbf{O_A}$ can be decomposed into 8 orthogonal primitive projection operators, with one of which we can project the real part of any element of $\mathbf{O}$, and twice that element minus the identity will yield the octonion conjugate.

$\mathbf{T_A}$ and Interpretation

If you're reading this, you're almost certainly familiar with the algebra

$$\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O},$$

which I demonstrated at the end of the last century is the mathematical blueprint for the Standard Model of quarks and leptons. This algebra is not a division algebra, and is not only noncommutative and nonassociative, it is also nonalternative. The algebra $\mathbf{T_A}$ is isomorphic to $\mathbf{C}(\mathbf{32})$ (32×32 matrices over $\mathbf{C}$). It is apparent, therefore, that the identity of $\mathbf{T_A}$ can be decomposed into 32 orthogonal primitive projection operators, with one of which we can project the complex part of any element of $\mathbf{T}$, but not the real part.

In each of these 4 cases, the elements of $\mathbf{K_A}$ are able to project from the elements of $\mathbf{K}$ a part that is either real or complex, and whichever mathematical field it is determines if the resulting algebra is fundamentally real or complex. This means that in creating a classical or quantum field theory over any of these spinor algebras, $\mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$ or $\mathbf{T}$, only if its underlying mathematical field is $\mathbf{C}$ can we hope to create a model that is consistent with our reality. I would suggest, therefore, that although quantum mechanics requires a division algebra, it makes no sense at all to use **H** or **O** alone, as they are functionally real. Tensoring with **C** renders either of them functionally complex. Tensoring all 3 yields a complex spinor space, allowing a complex quantization, and providing a framework for all the algebraic features of the Standard Model.

References:

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