

## Division Algebras: Family Replication

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27 June 2002

The link of the Division Algebras to 10-dimensional spacetime and one leptoquark family is extended to encompass three leptoquark families.

The origins of this work can be found in [1]. That book contains Lagrangians, interactions, and in general a more detailed development of the physics resulting from

$$\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$$

than has been presented elsewhere.

- $\mathbf{C}$  - complex numbers: associative, commutative, basis  $\{1, i\}$ ;
- $\mathbf{H}$  - quaternions: associative, noncommutative, basis  $\{1 = q_0, q_1, q_2, q_3\}$ ;
- $\mathbf{O}$  - octonions: nonassociative, noncommutative, basis  $\{1 = e_0, e_1, \dots, e_7\}$ ;

My subsequent work on these division algebras has been largely mathematical. Some of it deals with a more elegant derivation of the Standard Symmetry and lepto-quark family structure than is found in [1] (see [2,3]). This work accounts neatly for family structure, but it has not until now accounted for family replication.

- $\mathbf{K}_L, \mathbf{K}_R$  - the algebras of left and right actions of an algebra  $\mathbf{K}$  on itself.
- $\mathbf{K}_A$  - the algebra of combined left and right actions of an algebra  $\mathbf{K}$  on itself.
- $\mathbf{K}(m)$  -  $m \times m$  matrices over the algebra  $\mathbf{K}$ ;
- $\mathbf{K}^m$  - and  $m \times 1$  column over  $\mathbf{K}$ ;
- $\mathcal{CL}(p, q)$  - the Clifford algebra of the real spacetime with signature  $(p+, q-)$ .

If we let  $\mathbf{P} = \mathbf{C} \otimes \mathbf{H}$ , then  $\mathbf{P}_L$  is isomorphic to the Pauli algebra, so  $\mathbf{P}_L(2)$  is isomorphic to the Dirac algebra, and  $\mathbf{H}_R$ , which commutes with  $\mathbf{P}_L(2)$  (which acts on  $\mathbf{H}^2$ ), provides an internal  $SU(2)$  degree of freedom.

One can do much the same thing [1,2] with  $\mathbf{T}$ .  $\mathbf{T}_L$  is a Pauli-like algebra, and  $\mathbf{T}_L(2)$  is the Dirac algebra of 1,9-spacetime. Again there remains the internal  $\mathbf{H}_R$  commuting with  $\mathbf{T}_L(2)$ , providing an isospin  $SU(2)$ . The associated spinor space ( $\mathbf{T}^2$ ) transforms with respect to the standard symmetry as the direct sum of a leptoquark family and antifamily of 1,3-Dirac spinors.

But why should we need 2x2 matrices acting on  $\mathbf{T}^2$ ? And where are the other two families? To answer the second question I'll aggravate the first. In particular, we'll assume our spinor space is not just  $\mathbf{T}^2$ , but

$$\mathbf{C} \otimes \mathbf{H}^2 \otimes \mathbf{O}^3 = \mathbf{T}^6,$$

which is acted upon by  $\mathbf{T}_A(6)$ . (Octonion triples play important roles in many areas - derivations of the Leech lattice, the exceptional Jordan algebra, triality - which lends support to the idea that this expansion may be natural.)

Obviously, since  $\mathbf{T}^2$  accounts for one family/antifamily,  $\mathbf{T}^6$  would account for three, which is the accepted number of total families. However, in [2] the

algebra  $\mathbf{T}_L(2)$ , which acts on  $\mathbf{T}^2$ , is isomorphic to a Clifford algebra (the complexification of  $\mathcal{CL}(1, 9)$ ). Since all Clifford algebras are  $2^k$ -dimensional, the  $3^2 2^{13}$ -dimensional  $\mathbf{T}_A(6)$  (which is the full algebra of actions associated with  $\mathbf{T}^6$ ) is not a Clifford algebra.

Let's plow ahead anyway, and first look at the  $2^{15}$ -dimensional  $\mathbf{T}_A(4)$ , isomorphic to the complexification of  $\mathcal{CL}(1, 13)$ . This acts on  $\mathbf{T}^4$ , which is a pair of leptoquark families (and their antifamilies).

Let  $q_{Lk}$  and  $q_{Rk}$ ,  $k = 0, 1, 2, 3$ , be the respective left and right actions of the basis elements of  $\mathbf{H}$  on itself. Likewise,  $e_{La}$  and  $e_{Ra}$ ,  $a = 0, 1, \dots, 7$ , are the same for the octonions, although in this case, since  $\mathbf{O}_L = \mathbf{O}_R$ , we will not often be using the latter elements. Since the complex numbers are commutative and associative it makes no sense to distinguish left and right actions, so we won't.

Some  $2 \times 2$  real matrices:

$$\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Define, for example, the following  $4 \times 4$  real matrix:

$$[\beta \otimes \alpha] = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}.$$

Here is the chosen  $\mathcal{CL}(1, 13)$  1-vector basis:

$$[\epsilon \otimes \beta](iq_{R3}), [\epsilon \otimes \gamma]q_{Lk}e_{L7}(iq_{R3}), k = 1, 2, 3, [\epsilon \otimes \gamma]ie_{Lp}(iq_{R3}), p = 1, \dots, 6, \\ [\beta \otimes \epsilon]q_{R1}, [\beta \otimes \epsilon]q_{R2}, [\beta \otimes \alpha]q_{R3}, [\gamma \otimes \alpha].$$

The first line contains 10 elements which generate a  $\mathcal{CL}(1, 9)$  subalgebra of  $\mathcal{CL}(1, 13)$ . This is essentially the  $\mathcal{CL}(1, 9)$  that appeared in [2]. The second line contains 4 elements which generate a  $\mathcal{CL}(0, 4)$  subalgebra. Under the commutator product the associated 2-vectors are a basis for  $so(4) \sim su(2) \times su(2)$ . The six generators are:

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\{[\epsilon \otimes \alpha]q_{R1}, [\epsilon \otimes \alpha]q_{R2}, [\epsilon \otimes \epsilon]q_{R3}, \}.$$

The  $4 \times 4$  real matrix  $[\alpha \otimes \epsilon]$  is the product of the last four 1-vectors above, hence it commutes with the  $\mathcal{CL}(1, 9)$  1-vectors, but anticommutes with the  $\mathcal{CL}(0, 4)$  1-vectors. Therefore it can be used to reduce the 1,13-spacetime to 1,9-spacetime. In particular, at the 1-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])\mathcal{CL}(1, 13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = \mathcal{CL}(1, 9)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]).$$

At the 2-vector level,

$$\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])so(1, 13)\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]) = (so(1, 9) \times su(2))\frac{1}{2}(1 \pm [\alpha \otimes \epsilon]),$$

each projector  $\frac{1}{2}(1 \pm [\alpha \otimes \epsilon])$  picking out an  $su(2)$  half of  $so(4)$ , and projecting from the spinor space,  $\mathbf{T}^4$ , a  $\mathbf{T}^2$  subspace. Hence this reduction results in exactly the scenario developed in [2], except doubled. Each  $\mathbf{T}^2$  subspace is the direct sum of a family and antifamily of leptons and quarks.

With a Clifford algebra and spinors we can form a Dirac operator and Lagrangian. If there were  $2^k$  families then  $\mathbf{T}^{2k}$  would be the appropriate hyper-spinor space, acted on by a conventional Clifford algebra. But it is believed there are exactly 3 families, and we will have to get a little creative in constructing a Dirac-like Lagrangian for this case.

A Dirac operator for the  $\mathcal{CL}(1, 13)$  2-family model developed above would be

$$\begin{bmatrix} \partial_{1,9} & \partial_{0,4}^+ \\ \partial_{0,4}^- & \partial_{1,9} \end{bmatrix},$$

built from the original set of 14 1-vectors. For the 3-family case, the suggestion is to incorporate 3 of these 2-family Dirac operators, leading to a Lagrangian term like

$$\begin{bmatrix} \overline{\psi_1} & \overline{\psi_2} & \overline{\psi_3} \end{bmatrix} \begin{bmatrix} \partial_{1,9}^{a+} \partial_{1,9}^b & \partial_{0,4}^{a+} & \partial_{0,4}^{b-} \\ \partial_{0,4}^{a-} & \partial_{1,9}^{a+} \partial_{1,9}^c & \partial_{0,4}^{c+} \\ \partial_{0,4}^{b+} & \partial_{0,4}^{c-} & \partial_{1,9}^b \partial_{1,9}^c \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}$$

Each of the  $\psi_k, k = 1, 2, 3$ , is a complete leptoquark family plus antifamily. On the assumption this approach to a 3-family Lagrangian has some validity many questions arise. Are these the 3 observed families, or mixtures thereof? Are there 3 distinct 14-dimensional spaces? There are many more questions, which my intuition tells me are worth pursuing (no voices - just a gut feeling), but if this happens, it will do so slowly, as I didn't really have time to take it even this far.

## References:

- [1] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).
- [2] G.M. Dixon, [www.7stones.com/Homepage/10Dnew.pdf](http://www.7stones.com/Homepage/10Dnew.pdf)
- [3] G.M. Dixon, [www.7stones.com/Homepage/octoIII.pdf](http://www.7stones.com/Homepage/octoIII.pdf)