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Exponential Function

Some familiarity with algebraic concepts is assumed, but as this material will stress utility over rigor, there will be no theorems, axioms, corollaries or lemmas. So, when we say that A and B are elements of an algebra, we understand that they can be added $(A+B)$, subtracted $(A-B)$, and multiplied (AB), and in all cases the result is an element of the same set we started with. And if x is a real number, then xA is also an element of that set. Any generalizations of these ideas will occur as needed.

Let **R** be the algebra of real numbers, and **C** be the algebra of complex numbers. **C** is 2-dimensional over the reals: each complex number $z = x + iy$ has a real part and imaginary part (x and y are real numbers, so x is the real part of z, and iy the imaginary part).

In Lie group theory the exponential function is extremely important. The exponential of a real number is something one can easily determine by punching the number into a calculator and hitting the e^x button. The answer is a positive real number. But a calculator figures out this answer by expanding the exponential function in a Taylor series - and infinite polynomial sum (the calculator uses enough of the terms of the infinite sum to be as accurate as its screen allows). The Taylor expansion of $exp(A) = e^A$ is given below:

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + ...$$

If A is a real number, then the answer is a real number. But the righthand side of this equation can be defined for many other kinds of algebraic objects. This expression can be defined for any of the kinds of algebras we will deal with here. It uses algebraic addition, multiplication, and the multiplication by scalars (real numbers). Mathematicians worry about it being "well-defined", meaning that there is some unique algebraic answer to the infinite sum. We'll let them worry about that: all of our exponentials will be well-defined.

One of the most beautiful equations in all of mathematics is the exponential of a pure imaginary complex number:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \tfrac{\theta^2}{2!} - \tfrac{i\theta^3}{3!} + \tfrac{\theta^4}{4!} + ... \\ &= 1 - \tfrac{\theta^2}{2!} + \tfrac{\theta^4}{4!} + ...\{= cos(\theta)\} \\ &= i\theta - \tfrac{i\theta^3}{3!} + ...\{= isin(\theta)\} \\ &= cos(\theta) + isin(\theta). \end{aligned}$$

A really **big point** to make here is the following: the only property of the complex unit i used by this equation is $i^2 = -1$. If u is an element of an algebra that has an identity (1), and $u^2 = -1$, then $exp(u\theta) = cos(\theta) + usin(\theta)$. We shall use this fact repeatedly in what follows.

The norm squared (length squared) of the complex number $z = x + iy$ is: $zz^* = (x + iy)(x - iy) = x^2 + y^2$, where $z^* = (x - iy)$ is the complex conjugate of z. The complex conjugate of $exp(i\theta)$ is $exp(-i\theta) = cos(\theta) - isin(\theta)$, so the norm squared of $exp(i\theta)$ is

$$(e^{i\theta})(e^{i\theta})^* = (e^{i\theta})(e^{-i\theta}) = cos^2\theta + sin^2\theta = 1.$$

That is, the set of all $exp(i\theta)$ is just the unit sphere (circle) in the 2-dimensional complex plane. This set plays many roles. In particular, it is the 1-sphere, one of three nontrivial parallelizable spheres (more anon); and it is the Lie group $U(1)$.

Another example complicates matters a little: let u be as defined above, and let ρ be a projection operator belonging to the same algebra. That is, $\rho^2 = \rho$. Assume ρ commutes with u. (ρ is an idempotent by definition, but in refering to it as a projection operator stresses the idea that it acts on some space, which will almost always also be an algebra in these pages; also note that not all spaces have nontrivial (not equal 1 itself) projection operators - and in fact division algebras do not, else they would not be division algebras.) Consider the exponential $exp(u\rho\theta)$:

$$\begin{aligned}
e^{u\rho\theta} &= 1 + u\rho\theta + \tfrac{1}{2}(u\rho\theta)^2 + \tfrac{1}{6}(u\rho\theta)^3 + ... \\
&= 1 + u\rho\theta - \tfrac{1}{2}\rho(\theta)^2 - \tfrac{1}{6}u\rho(\theta)^3 + ... \\
&= 1 - \rho + \rho + u\rho\theta - \tfrac{1}{2}\rho(\theta)^2 - \tfrac{1}{6}u\rho(\theta)^3 + ... \\
&= 1 - \rho + \rho(1 + u\theta - \tfrac{1}{2}(\theta)^2 - \tfrac{1}{6}u(\theta)^3 + ... \\
&= (1 - \rho) + \rho e^{u\theta}.
\end{aligned}$$

Note that $(1 - \rho)^2 = (1 - \rho)$, so $(1 - \rho)$ is also a projection operator. Also, $\rho(1 - \rho) = \rho - \rho\rho = \rho - \rho = 0$. So ρ and $(1 - \rho)$ are orthogonal projection operators. In particular, ρ and $(1 - \rho)$ act on some space: they project subspaces, and these subspaces must be orthogonal. Because of the equation above it is clear that the action of $exp(u\rho\theta)$ on this space is nontrivial only on that subspace projected by ρ, while the subspace projected by $(1 - \rho)$ is left invariant. A good paradigm for all this is 2×2 real matrices, with $\rho = diag[1, 0]$:

$$exp(i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \theta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} e^{i\theta} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}.$$

Parallelizable Spheres

The set of all unit complex numbers (norm 1 and can be written in the form $exp(i\theta)$) is closed under multiplication ($exp(ia)exp(ib) = exp(i(a+b))$). This set fulfills all the requirements of a Lie group (roughly: multiplicative closure; inverses; identity; and the full set is a nice very smooth continuous set (manifold); we needn't go into elaborate rigorous detail - we'll come to know them when we see them). This Lie group is $U(1)$. The Lie algebra associated with $U(1)$ is the set from which $U(1)$ is derived via exponentiation. Hence the Lie algebra of $U(1)$, which we denote $u(1)$, is the set $\{i\theta : \theta \in \mathbf{R}\}$ - the set of purely imaginary complex numbers.

The associated manifold is the circle, the 1-dimensional sphere. This manifold has a special property: it is parallelizable (all Lie groups are, but only three spheres - although two of these three parallelizable spheres are also Lie groups - more anon). This can be visualized as follows: a manifold is parallelizable if it is possible to set all its points in smooth flowing motion at the same time (a good example of a nonparallelizable manifold is the 2-sphere (surface of a ball). There is always at least one point on the surface of a ball that is stationary: it's impossible to set them all in smooth motion at once (the fact that the earth has poles is related to this)). It must be further specified that at any point of the manifold the smooth flowing motion can flow in any direction. Clearly this is an area where mathematical rigor would be of some use, but our intention here is simply to introduce the concept as a way of highlighting the exceptional nature of the 1-, 3- and 7-spheres. Spin a 1-sphere and all the points move at once (not so a 2-sphere). It's clearly parallelizable.

The three hypercomplex division algebras are the complex numbers, **C** (2-d), quaternions, **H** (4-d), and octonions, **O** (8-d). Without getting too formal, these algebras are characterized by various properties, all of which boil down to a simple idea: they are all fundamentally very much like the complex numbers. And in particular, if A and B are elements of one of these three algebras, and $AB = 0$, then either $A = 0$ or $B = 0$. Clearly then there can not be any nontrivial projection operators, for if p (not equal to 1 (or 0, of course)) is a projection operator, then so is $(1-p)$, and $p(1-p) = p - pp = p - p = 0$. This contradicts the property above (no divisors of zero), hence p can not exist.

The other big property has to do with the norm. For any A and B in any of these algebras we can define norms (real lengths) $\|A\|$ and $\|B\|$, and these satisfy: $\|AB\| = \|A\|\|B\|$ (norm of the product is the product of the norms).

The subsets of norm = 1 of each of these three algebras are unit spheres: respectively, S^1 (1-sphere = circle); S^3 (3-sphere); and S^7 (7-sphere). Suppose U and V are elements of one of these spheres ($\|U\| = 1$; $\|V\| = 1$), then $\|UV\| = \|U\|\|V\| = 1$. So these spheres are closed under multiplication - and division (the multiplicative inverse of any A is $A^{-1} = A^*/\|A\|^2$).

Well, ok, these three spheres are closed under multiplication, they each possess an identity (clearly $\|1\| = 1$), and there are multiplicative inverses, so they seem to satisfy everything they'd need to satisfy to be Lie groups. And in fact the 3-sphere (unit quaternions) is a Lie group ($SU(2)$), but the 7-sphere (unit octonions) is not, even though it satisfies all of those nice Lie group properties, and it is parallelizable. The problem is, octonion multiplication is not associative, and in particular, if U, V and W are elements of the 7-sphere, then we can NOT in general conclude that $(UV)W = U(VW)$. And without that it's just not a Lie group (the fact that the position of the parentheses matters is extremely important, and in what follows we shall exploit this fact to create from the octonions lots of Lie groups - in particular, $SO(8)$ and $SU(3)$).

By the way, if U is a unit element of one of these three division algebras, and U is not the identity, then for any other unit element V, $UV \neq V$. That is, multiplication by U on the k-sphere (k=1,3 or 7) moves every element of the sphere (and smoothly). Hence the properties defining a division algebra imply these unit k-spheres are parallelizable. Of the three the 7-sphere is the only surprise: it is the only parallelizable manifold (with a suitably defined multiplication (I've forgotten the details)) that is not also a Lie group. And while we're at stressing the uniqueness of things, there are ONLY THREE hypercomplex division algebras (normed, with unity, and perhaps a condition or two more you'll have to look up), and this is equivalent to the fact that there are ONLY THREE parallelizable spheres, and this is equivalent to other unique properties in other mathematical realms. In particular, one can construct spheres of other dimensions, but none will be parallelizable and none associated with a division algebra; and one can construct algebraic generalizations of **C**, **H** and **O**, but none will yield more division algebras nor be associated with other parallelizable spheres.

We have here a closed context of extremely unique and generative mathematical objects (generative?: for example, all the classical Lie groups are associated with **R**, **C**, **H** or **O** - **R** is also a division algebra; its associated unit sphere consists of two points, +1 and -1). I personally do not believe that there is much to gain by generalizations, for they do not carry with them the myriad associated special properties. Mathematics seems to resonate at these special dimensions (1,2,4,8), and no others in the same way or to the same extent (although we will look at 24 as well, a dimension that resonates in a different way (for an introduction to that, look at "Sphere Packings" by Conway and Sloane)).

Quaternions: Lie Groups, Clifford Algebras and Spinors

Let q_j, j = 1,2,3, represent the pure imaginary quaternion units (let $q_0 = 1$). The multiplication table of the pure quaternion units is cyclic ($q_1 q_2 = q_3 \Rightarrow q_2 q_3 = q_1 \Rightarrow q_3 q_1 = q_2$). Add to this the fact that these units anticommute and you've got the whole multiplication table. It's interesting to represent these units by matrices, and our first representation will be by real 4x4 matrices. For example:

$$q_1 \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_1 \\ -q_0 \\ q_3 \\ -q_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Note: because the **H** imaginary units anticommute, multiplication by this same element from the right will yield a different matrix (the lower right block is the negative of that given above). So, we have the identity from q_0 (left or right multiplication), three matrices by left multiplication of imaginary units, and three by right multiplication. And then there are 9 = 3x3 from simultaneous left/right multiplication by imaginary units (indices = 1,2,3). That makes 1+3+3+9 = 16 in all, and in fact these 16 matrices form a basis for the 16-dimensional real algebra of 4x4 real matrices (**R**(4)).

If you bother to compute the 3 left multiplication matrices, and 3 right multiplication matrices, you will find that the left (or right) multiplication matrices anticommute with one another, but each of the left multiplication matrices commutes with each of the right multiplication matrices. The adjoint algebra of left actions of **H** on itself commutes with the adjoint algebra of right actions (as we will see, this is related to the following Lie algebra identity: $so(4) = su(2) \times su(2)$; i.e., the 6-dimensional Lie algebra $so(4)$ consists of two commuting copies of the 3-dimensional $su(2)$).

The fact that **H** acting on itself from the left and from the right gives rise to two distinct and commuting copies of **H** actions suggests that it would be worth our while to distinguish the algebras of left actions, right actions, and **H** itself, the algebra on which these adjoint algebras act:

- **H**: quaternion algebra itself; basis q_m, m=0,1,2,3;
- $\mathbf{H}_L$: adjoint algebra of left actions of **H** on itself; basis q_{Lm}, m=0,1,2,3;
- $\mathbf{H}_R$: adjoint algebra of right actions of **H** on itself; basis q_{Rm}, m=0,1,2,3.

Why bother with three copies of the same algebra? Because the quaternions are noncommutative, and there really are three different copies. Using all three makes it very easy to connect the quaternions to some important Lie groups, Clifford algebras and spinors.

Any pure imaginary quaternion, A, can be written in the form $u\theta$, where u is a unit imaginary quaternion (so an element of a 2-sphere; don't forget, the space of imaginary quaternions is 3-dimensional), and $\theta = \|A\|$, the positive real magnitude of A. The element u behaves just like the complex imaginary i when exponentiated, because $u^2 = -1$. Therefore, $e^A = exp(u\theta) = cos(\theta) + usin(\theta)$. This is also a unit quaternion, although not a pure imaginary one. In fact, any unit quaternion can be written in this form. Hence the set $\{e^A : A$ linear in $q_k, k = 1, 2, 3\} = \{U \in \mathbf{H} : \|U\| = 1\} = S^3$ = 3-sphere.

This set is also closed under multiplication, and since it is associative, it is a Lie group, in this case $SU(2)$ (that is, the "shape" of $SU(2)$ is that of a 3-sphere). The associated Lie algebra, $su(2)$, has a basis, q_k, k = 1,2,3 (3-dimensional, as is $SU(2)$ itself). That is, $SU(2)$ is obtained from the elements of $su(2)$ via exponentiation. Note that a Lie algebra by definition uses the commutator product, under which the set of all elements linear in q_k, k = 1,2,3, is closed.

However, Lie groups invariably appear in physics as actions on some space, not as some abstract mathematical object unconnected to anything else. In order to make connection with those ideas we have to start using the adjoint algebras.

Let A be a pure imaginary quaternion as on the previous page, so $U = e^A$ is an element of $SU(2)$ = 3-sphere. Suppose this $SU(2)$ acts on some space, M. Well, does U act on M from the left or right? It can do either, and it matters. M has a copy of $\mathbf{H}$ in it, and it's this copy that receives the action of U.

Given $A = A^k q_k$, sum k=1,2,3,

define $A_L = A^k q_{Lk}$, and $U_L = exp(A_L)$;

define $A_R = A^k q_{Rk}$, and $U_R = exp(A_R)$.

So $U_L[M] = UM$, and $U_R[M] = MU$, and both of these are $SU(2)$ actions, but they're distinct $SU(2)$'s, and the left action $SU(2)$ commutes with the right action $SU(2)$ (because $\mathbf{H}$ is associative).

By the way, using U_L and U_R we can construct a copy of $SO(3)$, the automorphism group of $\mathbf{H}$. In particular, if $X \in \mathbf{H}$, then $U_L U_R^{-1}[X] = UXU^{-1}$ leaves the real part of X alone and performs an $SO(3)$ rotation on the imaginary 3-dimensional part (the reader should see this action is obviously a $\mathbf{H}$ automorphism).

$\mathbf{H}$ is also a Clifford algebra, and an integral part of Clifford algebra theory. However, Clifford algebras also act on some space (which are called spinor spaces), so we should once again specify the direction of action. Let $\mathcal{CL}(p, q)$ be the Clifford algebra of the real pseudo-orthogonal space with signature p(+), q(-). Then Q_L is isomorphic to $\mathcal{CL}(0, 2)$, a 1-vector basis being $\{q_{L1}, q_{L2}\}$, and the sole 2-vector basis element: $q_{L3} = q_{L1} q_{L2}$. Likewise $\mathbf{H}_R$ is isomorphic to $\mathcal{CL}(0, 2)$.

What if we allow elements of both $\mathbf{H}_L$ and $\mathbf{H}_R$? We'll denote by $\mathbf{H}_A$ the algebra of combined left/right actions of $\mathbf{H}$ on itself. This algebra is isomorphic to $\mathbf{R}(4)$ (hence also to $\mathcal{CL}(3, 1)$ and $\mathcal{CL}(2, 2)$). But this is a path down which I haven't the patience at present to trod. Time for octonions.

Octonions: Lie Groups, Clifford Algebras and Spinors

The octonion units will be denoted e_a, a = 0, 1, 2, ...,7, $e_0 = 1$, the identity, and the most commonly used octonion multiplication table for the remaining 7 units is determined by $e_a e_{a+1} = -e_{a+1} e_a = e_{a+3}$, where these values cycle through the set {1,2,3,4,5,6,7}.

Like we did for the quaternions, if A is an imaginary octonion (no real part), then $A = u\theta$, where $\|u\| = 1$, and $\|A\| = \theta$, so u is an element of the imaginary octonion unit 6-sphere and is the direction of A, and θ is the magnitude of A. Again, $u^2 = -1$, so u behaves like i when exponentiated, and $e^A = e^{u\theta} = cos(\theta) + usin(\theta)$. The set of unit octonions = $\{e^A : A \text{ imaginary}\} = S^7$ = 7-sphere. Again, since $\|XY\| = \|X\|\|Y\|$, X and Y arbitrary octonions, this 7-sphere is closed under multiplication, and it is easily shown to be parallelizable. But it is NOT a Lie group, because it is not associative, and hence it can NOT be represented by a matrix algebra.

But there are real matrices associated with the octonions, and lots of Lie groups. (Note: I said these matrices are associated with the octonions, not that they represent the octonions. I've seen lots of papers claiming to represent the octonions by real matrices - or complex. Can't be done. There are associations that can be made, but real and complex matrix algebras are necessarily associative, hence whatever algebra results is not the octonion algebra - it's something else. Don't be fooled by substitutes - look for the real octonion label. Of course, if you adopt a multiplication other than conventional matrix multiplication, anything can be done, and I do this elsewhere on this site, but my motivation is pure, and the result beautiful.)

As we did with the quaternions, we can make some obvious associations:

$$e_1 \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} e_1 \\ -e_0 \\ e_4 \\ e_7 \\ -e_2 \\ e_6 \\ -e_5 \\ -e_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}$$

$$e_2 \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} e_2 \\ -e_4 \\ -e_0 \\ e_5 \\ e_1 \\ -e_3 \\ e_7 \\ -e_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix}$$

There is a unique matrix derived in this way for each of the 8 basis units (including the identity). Like the imaginary octonion units themselves, the 7 matrices associated with the imaginary octonion basis units anticommute, but unlike the octonion units, these matrices do not close under multiplication. In fact, via matrix products and sums they generate all of $\mathbf{R}(8) = \mathcal{CL}(0,6)$, a 64-dimensional algebra. However, although the product of the two matrices above may not be associated with a single octonion unit, it is in fact associated with a more complicated octonion action arising out of octonion nonassociativity.

For X an arbitrary octonion, the first matrix is associated with the action, $e_{L1}[X] = e_1X$, the second matrix with the action $e_{L2}[X] = e_2X$, and the product of the second matrix times the first with the action: $e_{L12}[X] = e_1(e_2X)$. (Ignore the fact that the matrix product is associated with the reverse octonion product; what's important is that the product is in fact associated with this embedded action involving two octonion units; we won't be using the matrices after this.)

Define the general left embedded adjoint action of **O** on itself by

$$e_{Lab...c}[X] = e_{La}e_{Lb}...e_{Lc}[X] = e_a(e_b(...(e_cX)...)).$$

It can be shown that

$$e_{L123456}[X] = e_{L7}[X]$$

for $X \in \mathbf{O}$. Also, for distinct $a, b \in \{1, 2, ..., 7\}$,

$$e_{La}e_{Lb} = e_{Lab} = -e_{Lba} = -e_{Lb}e_{La}.$$

Also, $e_{La}e_{La} = -1$. All together this implies that a complete basis for the algebra of left action of **O** on itself ($\mathbf{O}_L$) consists of:

the identity, 1;

the 7 distinct e_{La};

the 21 distinct e_{Lab};

and the 35 distinct e_{Labc},

where $a, b, c \in \{1, 2, ..., 7\}$.

Consequently $\mathbf{O}_L$ is 64-dimensional, and it is isomorphic to $\mathbf{R}(8) = \mathcal{CL}(0,6)$. The spinor space of $\mathbf{R}(8) = \mathcal{CL}(0,6)$ is the space of 8×1 real column matrices; the spinor space of $\mathbf{O}_L = \mathcal{CL}(0,6)$ is **O** itself. In both cases the spinor space is 8-dimensional.

A 1-vector basis for $\mathbf{O}_L = \mathcal{CL}(0,6)$ is $e_{Lp}, p = 1,...,6$ (as we saw above, the 7-vector basis is the element e_{L7}). The 2-vector basis of $\mathcal{CL}(p.q)$ can be thought of as the Lie algebra $so(p,q)$, hence

$$so(0,6) = so(6) \longrightarrow \{e_{Lpq}, p, q = 1, ..., 6\}.$$

Two other Lie algebras while we're at it:

$$so(7) \longrightarrow \{e_{Lab}, a, b = 1, ..., 7\};$$

$$so(8) \longrightarrow \{e_{Lab}, e_{Lc}, a, b, c = 1, ..., 7\}.$$

The automorphism group of the octonions is the exceptional Lie group G_2, and we'll denote its Lie algebra LG_2. As was shown in the book, if a,b,c,d are distinct indices from 1 to 7, and $e_a e_b = e_c e_d$, then $e_{Lab} - e_{Lcd} \in LG_2$, and the space spanned by elements of this form is 14-dimensional.

For example, let $u = \frac{1}{2}(e_{L56} - e_{L37})$. Using the fact that $e_{L7654321} = 1$, we can show that $u^2 = -p = -\frac{1}{2}(1 - e_{L421})$, and $u^3 = -pu = -u$. Note that $u^4 = p^2 = -puu = -uu = p$, so p is a projection operator. All this implies

$$e^{u\theta} = (1-p) + p(cos(\theta) + usin(\theta)) \in G_2.$$

Therefore, $e^{u\pi} = (1-p) - p = e_{L421}$. This has the following action on the basis of $\mathbf{O}$: $e_{L421}[e_k] = e_k, k = 0, 1, 2, 4$, and $= -e_k, k = 3, 5, 6, 7$. This is a sign changing automorphism, and there's one for each of the 7 quaternionic triples of octonion units.

Finally, if p,q,r,s are distinct indices from 1 to 6 (note: 7 is singled out for both historical and mathematical reasons), and $e_p e_q = e_r e_s$, then $e_{Lpq} - e_{Lrs} \in su(3)$, a subalgebra of LG_2. This Lie algebra generates a subgroup of G_2 that leaves e_7 invariant (obviously there is a copy of $SU(3)$ in G_2 for every imaginary octonion direction, so $G_2/SU(3)$ is the 6-sphere with opposite points identified).

One of the more interesting consequences of octonion nonassociativity is $\mathbf{O}_L = \mathbf{O}_R$. That is, if we define $e_{Rab...c}[X] = (...((Xe_a)e_b)...)e_c$, then the algebra of actions spanned by elements of that form ($\mathbf{O}_R$) is exactly the same as $\mathbf{O}_L$ (recall that while $\mathbf{H}_L$ and $\mathbf{H}_R$ are isomorphic, they are in fact distinct). It should not be surprising that $\mathbf{O}_L = \mathbf{O}_R$, for the spinor space of both is 8-dimensional $\mathbf{O}$, and as $\mathbf{O}_L$ is isomorphic to $\mathbf{R}(8)$, there are no actions on the spinor space not accounted for by $\mathbf{O}_L$.

One can connect the two adjoint algebras in the following way: if a,b,c,d,r,s,u,v are 8 distinct indices from 0 to 7, and $e_a e_b = e_c e_d = e_r e_s = e_u e_v$, then

$$e_{Lab} = \frac{1}{2}(-e_{Rab} + e_{Rcd} + e_{Rrs} + e_{Ruv}),$$

$$e_{Rab} = \frac{1}{2}(-e_{Lab} + e_{Lcd} + e_{Lrs} + e_{Luv}).$$

(Note: one of these indices must be 0; it can be deleted from these equations.) Consequently, if none of the indices a,b,c,d is 0, then $e_{Lab} - e_{Lcd} = -e_{Rab} + e_{Rcd}$. That is, LG_2 (and so also $su(3)$) looks very much the same in $\mathbf{O}_R$ as it does in $\mathbf{O}_L$.