Mathematical Restrictions on the Multiverse

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Theoretical physics, and in particular the search for a Theory of Everything, has for some time been approaching a barrier to advancement perhaps no less formidable than the speed of light is to relative velocity, or a temperature of absolute zero is to thermodynamics. In those cases the energy needed to achieve the limit rises arbitrarily the closer to the limit one gets. Advancement to a Theory of Everything, on the other hand, is increasingly hindered on both the mathematical and experimental fronts by the need for energies of a different sort: intellectual and financial. As a consequence theory and experiment - historically each contributing significantly to the other's advancement - are drifting apart. To be sure, experimental verification is not required for theoretical advancement, but unchecked advances run the risk of being overly influenced by fashion (frequently linked to funding) and personal bias. Predictions made in such equivocal circumstances are as apt to be discussed dramatically in the popular literature as soberly in technical journals.

And that brings us to the multiverse, and in particular the notion that not only is our perceived universe but one of many, but that the others may have bizarrely different properties. If a theory is correct, and it does not preclude a thing, then that thing is a prediction, dependent upon constraints and circumstances. If a theory allows for Flatland, then a Flatland universe may exist outside of our own.

A great hope of many theorists is that an Idea will come along (preferably an extension of the one they're working on) that will somehow lead to the inevitability of our universe, of its particles, dimensions and interactions, excluding Flatland and other unobserved variants. Although not a String Theory acolyte myself, recent readings make it apparent that lately that hoped for inevitability - at least in the context of this most fashionable approach to unification - has taken several powers of ten steps backward.

This is a disappointment to those in the field, but it doesn't mean String theory is wrong, or even that a surfeit of solutions is a sign of a problem. We can always adjust our perspective, made all the easier in this case given the absence of any way to verify the rightness or wrongness of any perspective we may wish to adopt. And anyway M-theory is waiting in the wings, and order may ultimately once again be restored.

Meanwhile we are led to an almost Zen-like contemplation of the multiverse, and the possibility that ours is but one of many universes, and its characteristics a matter of chance (and lucky for us). Some may relish this notion - hell, if it's true I'd relish it, as I would relish the arrival of extraterrestrials (friendly ones). However, let's not be too hasty.

There are indications from mathematics that there may be restrictions to the kinds of universes any hypothetical multiverse may contain. First we have to accept that mathematics is in large part pan-universal. Clearly if we become able to construct a Theory of Everything based on mathematics, and that Theory predicts other viable universes with different physics, then that same mathematical foundation will work in those other universes to imply the potential for the existence of our universe, and of us, which I find personally gratifying.

In our mathematics there are what I have termed resonances, dimensions in which things just seem to fall into place, where there is a wealth of interesting structures and symmetries absent or lesser in all other dimensions. These are the dimensions 1, 2, 4, 8 and 24.

To illustrate this resonant quality we'll look at spheres. In a Euclidean space of dimension k, the set of all points a fixed distance from a given point is the (k-1)-sphere. In 3-space (where we live) the surface of a ball is a 2-sphere, each point being the same distance from the ball's center. In 2-space (a piece of paper) we get the 1-sphere, which is just a circle. Such a circle exists in 2-space but is a 1-dimensional object (a curved line). Likewise the surface of a ball exists in 3-space but is 2-dimensional.

There is another very important distinction between the 1-sphere and 2-sphere: if you cover a 2-sphere with an oil slick, it is impossible to set every molecule of the oil slick in linear motion at the same time. There will always be at least one dead point where there can be no molecular motion save rotation. For this reason it is impossible to cover the earth with a grid system (like longitude and latitude) that doesn't have poles. At the poles the grid becomes in-determinant in one dimension (what's the longitude at the North Pole?). The 1-sphere on the other hand doesn't have this problem. We have a common grid on the 1-sphere that fixes each point uniquely: degrees. (In fact, we can add multiples of 360 degrees and be at the same point, but that's not the same as having no definable value at all, like the longitude at the poles). Likewise, put an oil slick around a circle and every molecule can be set in motion at the same time. This property is called parallelizability.

Interestingly, and somewhat astoundingly, there are only four values of k for which the (k-1)-sphere is parallelizable: k = 1, 2, 4 and 8. Further, for each of these values of k the parallelizability of the (k-1)-sphere implies the existence of a division algebra. These are, respectively, the real numbers, **R**, complex numbers, **C**, quaternions, **H**, and octonions, **O** (over the real numbers these algebras have respective dimensions 1, 2, 4 and 8). And the existence of these division algebras gives rise to myriad other special objects. (There are different ways to extend the sequence of division algebras to higher dimensions, and it was to forestall interest in these higher dimensional variants that I chose to introduce these algebras via the finite sequence of parallelizable spheres. No other dimension, save 24 (for different reasons), is as rich in interesting mathematical structures. It is this resonant richness that is important, not the extendibility of one aspect of this richness by any concocted means.)

What does any of this have to do with physics? Well, before proceeding I should explain that unlike Eugene Wigner, a founder of modern quantum theory, I do not at all find the effectiveness of mathematics in the natural sciences in any way unreasonable. In fact, I believe that at their most profound depths mathematics and theoretical physics are the same. This kind of thinking probably belongs to a school of thought founded by a dead Greek or two, but be that as it may, it is this belief that led me to assume that it would be impossibly surprising if physical reality and resonant mathematics were not found to be inextricably intertwined.

Early in the last century P.A.M. Dirac managed to unify the ideas of Special Relativity and Quantum Mechanics. The resulting Relativistic Quantum Mechanics makes heavy use of matrix mathematics, and a model of the electron evolved in which it was represented mathematically by a column matrix of 4 variable complex numbers. This is called a Dirac spinor, and it is naturally associated with 1,3-spacetime. It is frequently useful, however, to consider the Dirac spinor as being built from two Pauli spinors (columns of 2 complex numbers), for it turns out the electron behaves differently depending on certain properties that vary at the level of these Pauli spinor parts of the whole Dirac spinor. There are infinitely many kinds of spinors, but the Pauli and Dirac spinors are those associated with our space-time and our particles.

Interestingly each of the four division algebras, **R**, **C**, **H** and **O**, can also be viewed as a spinor space. And this is also true of combinations of the division algebras. For example, the complexification of the quaternions (denoted $\mathbf{P} = \mathbf{C} \otimes \mathbf{H}$) can be viewed as a pair of Pauli spinors, and if we form a column matrix of two such elements we get a pair of Dirac spinors. Moreover, the fact that they come in pairs is a good thing, because elementary particles also come in pairs: the electron is paired with the electron neutrino; the up-quark with the down-quark; and so on. The mathematics linking these pairs is the Lie group SU(2). It is very interesting that the set of all unit quaternions is a copy of SU(2) (since **H** is 4-dimensional, the set of unit quaternions is topologically the same as the set of all points in 4-space a distance 1 from the origin, which is the 3-sphere, one of our four parallelizable spheres).

The notion that the 1,3-spacetime we perceive ourselves to live in may be but a part of some higher dimensional spacetime has been knocked around for many years, but String theory's 1,9-spacetime has received more attention than the alternatives. Surprisingly this same spacetime arises from the mathematics of the division algebras. The algebra $\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$ (the complexification of the quaternionization of the octonions ... it's actually a very simple joining of all three into one big algebra, each doing its own thing) bears the same relationship to 1,9-spacetime that $\mathbf{P} = \mathbf{C} \otimes \mathbf{H}$ bears to 1,3-spacetime. **T** is a kind of Pauli spinor for 1,9-spacetime, and doubling it gives rise to a Dirac spinor for 1,9-spacetime.

There are natural symmetries (Lorentz groups) associated with these spinors and spacetimes, and it is very interesting what happens to the symmetry of the 1,9-spacetime symmetry when we mathematically reduce the 1,9-spinors to 1,3-spinors. As expected, the 1,9-Lorentz group reduces to the 1,3-Lorentz group, but two other bits of the 1,9-Lorentz group also survive: a U(1) piece; and an SU(3) piece. The **P** and **T** spinors are already SU(2) doublets, so that leaves us with the reduced group, SO(1,3) x U(1) x SU(2) x SU(3).

This boils down to something pretty simple: the pieces of this reduced group are just those needed to describe the leptons and quarks of the Standard Model of elementary particle theory. What's more, the resulting bits of the reduced **T**-spinor behave mathematically just like a family of leptons (electron and its neutrino) and quarks (up and down), together with its antifamily.

Granted, this result is not purely mathematical - some physics finds its way into the derivation. The choice was made to reduce the 1,9-spinors to 1,3-spinors because we physically experience a 1,3-spacetime. On the other hand, **P** is associated with 1,3-spacetime, and it is one division algebra short of **T**, associated with 1,9-spacetime, so one could also argue in this and other ways that this reduction is mathematically natural.

Whatever the case, and whether or not one accepts that the form of our universe has anything to do with this remarkable mathematics/physics correspondence, it is undeniable that much of what we have come to accept about the mathematical nature of our physical reality is very precisely mirrored by these resonant mathematical structures. Were we to hypothesize the existence of other universes with other physics, the resonant mathematics of the division algebras would not change in any of these alternate universes, and it would still mirror our universe, with its particular physics. How far then can we be justified in postulating variations? If our theories predict a multiverse populated with universes with widely diverging physics, how strange it would be that we happen to exist in a variant that so neatly meshes with the mathematics of the resonant dimensions, 1, 2, 4 and 8 (the final resonant dimension may also play a role, connected to the fact that we live in a universe with not just one family of leptons and quarks, but three)? I suggest that it would be so strange as to be incredible, and that theories that are overly profligate with their multiverse predictions may be ruled out as a consequence.

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