C, O, Split O.

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Abstract

In response to a request, here using the split octonions to generate the Clifford algebra, $\mathcal{CL}(3,3)$, some Lie algebras, including a split G_2.

"The object of life is not to be on the side of the majority, but to escape finding oneself in the ranks of the insane." Marcus Aurelius

A quick reminder: if you want to understand the notation you'll have to look in older articles and the book.

The algebra $\mathbf{D} = \mathbf{C} \otimes \mathbf{O}$ is also the spinor space of the algebra of left actions of this algebra on itself: $\mathbf{D}_L = \mathbf{C}_L \otimes \mathbf{O}_L$. Since the quaternion algebra is not included in this tensor product, the algebra $\mathbf{O}_L = \mathbf{O}_R$, and $\mathbf{C}$ is commutative, we note in that

$$\mathbf{D}_L = \mathbf{D}_R \simeq \mathcal{CL}(7,0).$$

An obvious basis of 1-vectors for this Clifford algebra is the set

$$\{ie_{La}, a = 1, ..., 7\}.$$

From these we generate the 21 dimensional space of 2-vectors, a Lie algebra with respect to the commutator product:

$$\{e_{Lab}, a \neq b = 1, ..., 7\}.$$

Given the index cycling and doubling invariant octonion product arising from the quaternion triple, $e_1 e_2 = e_4$, we can generate the following 8-dimensional subalgebra of the 16-dimensional $\mathbf{D}$:

$$\hat{\mathbf{O}} = \text{span}\{1 = e_0, e_1, e_2, ie_3, e_4, ie_5, ie_6, ie_7\}.$$

This is an example of a split octonion algebra (not the only copy in $\mathbf{D}$). $\mathbf{O}$ by itself is the spinor space of $\mathbf{O}_L = \mathbf{O}_R \simeq \mathcal{CL}(0,6)$. $\hat{\mathbf{O}}$ is the spinor space of

$$\hat{\mathbf{O}}_L = \hat{\mathbf{O}}_R \simeq \mathcal{CL}(3,3)$$

(the left hand equality may not be obvious, but it is not difficult to demonstrate).

The obvious 1-vector basis of this copy of $\hat{\mathbf{O}}_L \simeq \mathcal{CL}(3,3)$ is

$$\{e_{L1}, e_{L2}, e_{L4}, ie_{L3}, ie_{L6}, ie_{L5}\},$$

and from this we get the following basis for the Lie algebra $so(3,3)$:

$$\begin{aligned}&\{e_{L12}, e_{L24}, e_{L41}, e_{L63}, e_{L56}, e_{L35},\\ &ie_{L13}, ie_{L16}, ie_{L15}, ie_{L23}, ie_{L26}, ie_{L25}, ie_{L43}, ie_{L46}, ie_{L45}\}.\end{aligned}$$

The compact Lie subalgebra (first line) is $so(3) \times so(3) \simeq su(2) \times su(2)$.

By including ie_L7 in the mix we can generate the lie algebra $so(3,4)$:

$$\begin{aligned}&\{e_{L12}, e_{L24}, e_{L41}, e_{L63}, e_{L56}, e_{L35}, e_{L57}, e_{L67}, e_{L37},\\ &ie_{L13}, ie_{L61}, ie_{L15}, ie_{L23}, ie_{L26}, ie_{L52}, ie_{L34}, ie_{L46}, ie_{L45}, ie_{L71}, ie_{L72}, ie_{L74}\}.\end{aligned}$$

Without the complex i (that is, ordinary octonions), we would have arrived at this point with a copy of $so(7)$ instead of $so(4,3)$. The Automorphism group of $\mathbf{O}$ is G_2, a 14-dimensional subgroup of $SO(7)$. Its Lie algebra has a basis:

$$LG_2 \longrightarrow \{(e_{Lab} - e_{Lcd}) : \text{distinct } a, b, c, d = 1, ..., 7, e_{Lab}[1] = e_{Lcd}[1]\}.$$

Now we go back to $\hat{\mathbf{O}}$ and from $so(4,3)$ arrive in similar fashion at a basis for a split copy of LG_2:

$$\begin{array}{ll}
e_{L12} - e_{L63} & i(e_{L13} - e_{L26}) \\
e_{L24} - e_{L56} & i(e_{L26} - e_{L43}) \\
e_{L41} - e_{L35} & i(e_{L61} - e_{L23}) \\
e_{L12} - e_{L57} & i(e_{L52} - e_{L46}) \\
e_{L24} - e_{L37} & i(e_{L34} - e_{L15}) \\
e_{L41} - e_{L67} & i(e_{L61} - e_{L74}) \\
 & i(e_{L52} - e_{L71}) \\
 & i(e_{L34} - e_{L72})
\end{array}$$

Again, the compact generators (left) form a basis for the Lie subalgebra $su(2) \times su(2)$.

And now I'll stop until inspiration and time prompt me to carry on. By the way, although I've done very little with the split octonions over the years, I first encountered them in the work of Gürsey and Günaydin in the mid 1970s. (I don't have time to proof-read this at the moment; please let me know of any egregious errors (gdixon@7stones.com, subject "split octonions").)

References

[1] www.7stones.com

[2] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Springer Verlag).