

# Octonions: $E_8$ Lattice to $\Lambda_{16}$ .

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## Abstract

I present here another example of a lattice fibration, a discrete version of the highest dimensional Hopf fibration:  $S^7 \rightarrow S^{15} \rightarrow S^8$ .

## 1. $D_4$ to $E_8$ with Quaternions.

To motivate the higher dimensional case involving the octonions, I'll first develop a lower dimensional lattice fibration using the quaternions.

Let  $q_m$ ,  $m = 0, 1, 2, 3$ , be a conventional basis for the quaternion algebra [1],  $\mathbf{Q}$ . Define

$$D_4^+ = \{\pm q_m\} \cup \left\{ \frac{1}{2}(\pm q_0 \pm q_1 \pm q_2 \pm q_3) \right\}. \quad (1)$$

These  $8 + 16 = 24$  unit quaternions form the inner shell (nearest neighbors to the origin) of a  $D_4 = \Lambda_4$  lattice ( $\Lambda_k$  is the real laminated lattice in  $k$  dimensions [2]). It is well known that the set  $D_4^+$  is closed under multiplication.

Define

$$D_4^- = \left\{ \frac{1}{2}(\pm q_m \pm q_n) \right\}. \quad (2)$$

This is also the inner shell of a  $D_4$  lattice, these elements normalized to  $1/\sqrt{2}$ . This is a (shrunk)  $Spin(4)$  rotation of  $D_4^+$ . However,  $D_4^-$  is not closed under multiplication even if expanded to the unit sphere.

From these two sets we can construct the inner shell of the 8-dimensional  $E_8 = \Lambda_8$  lattice. In particular,

$$\begin{aligned} & \{ \langle U, 0 \rangle, \langle 0, V \rangle : U, V \in D_4^+ \} \quad (2 \times 24 = 48 \text{ elements}) \\ \cup & \{ \langle U, V \rangle : U, V \in D_4^-, UV^\dagger = \pm q_m \} \quad (8 \times 24 = 192 \text{ elements}) \end{aligned} \quad (3)$$

( $m \in \{0, 1, 2, 3\}$ ) is the inner shell of an  $E_8$  lattice, a subset of the unit 7-sphere in  $\mathbf{Q}^2$ .

To illustrate the second of the sets in (3), lets look at the set of all  $V \in D_4^-$ ,  $UV^\dagger = \pm q_m$  for  $U = \frac{1}{2}(1 + q_1)$ . There are 8 such elements:

$$\pm \frac{1}{2}(1 + q_1); \quad \pm \frac{1}{2}(1 - q_1); \quad \frac{1}{2}(\pm q_2 \pm q_3). \quad (4)$$

In fact, there are 8 such elements for each  $U \in D_4^-$ , hence that second set has  $8 \times 24 = 192$  elements. The total number of elements is  $48 + 192 = 240$ , which is the order of  $E_8$  [2].

There is another characterization of this 192 element subset:

$$\{ \langle U, V \rangle : U, V \in D_4^-, V = \pm U \text{ or } U + V \in D_4^+ \}. \quad (5)$$

The first two elements in (4) are  $\pm U$ , and the remaining six elements satisfy  $U + V \in D_4^+$ .

## 2. Fibrations.

If  $U, V \in \mathbf{Q}$  satisfy

$$UU^\dagger + VV^\dagger = 1,$$

then the doublet  $\begin{bmatrix} U \\ V \end{bmatrix}$  is an element of  $S^7$ , the (unit) 7-sphere. Define the map

$$\begin{aligned} \begin{bmatrix} U \\ V \end{bmatrix} &\rightarrow \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}^\dagger = \begin{bmatrix} UU^\dagger & UV^\dagger \\ VU^\dagger & VV^\dagger \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned} \quad (6)$$

$$+ \frac{UU^\dagger - VV^\dagger}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{UV^\dagger + VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{UV^\dagger - VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set of all elements

$$\left\langle \frac{UU^\dagger - VV^\dagger}{2}, \frac{UV^\dagger + VU^\dagger}{2}, \frac{UV^\dagger - VU^\dagger}{2} \right\rangle$$

(first two real, third pure quaternion, so 5-dimensional) covers  $S^4$  (the 4-sphere in  $\mathbf{R}^5$ , in this case of radius  $\frac{1}{2}$ ). The map (6) is an example of the sphere fibration [1]

$$S^7 \xrightarrow{S^3} S^4. \quad (7)$$

(Another more interesting example of this fibration in terms of the octonions was given in [3].)

If  $\begin{bmatrix} U \\ V \end{bmatrix} \in E_8$ , as defined in (3), then the map (6) takes  $E_8$  onto the lattice  $Z^5$  (all inner shells at this point), consisting of elements of the form

$$\begin{aligned} &\frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &\pm \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \pm \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \pm \frac{q_i}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned} \quad (8)$$

( $i = 1, 2, 3$ ). This is an example of the lattice fibration

$$E_8 \xrightarrow{D_4} Z^5. \quad (9)$$

(Again, in [3] a more interesting example of this fibration was presented using the octonions.)

### 3. $E_8$ to $\Lambda_{16}$ with Octonions.

Let  $\mathbf{O}$  be the octonion algebra [1,3,4]. I choose an octonion multiplication whose quaternionic triples are determined by the cyclic product rule,

$$e_a e_{a+1} = e_{a+5}, \quad a \in \{1, \dots, 7\}, \quad (10)$$

where the indices in (4) are from 1 to 7, modulo 7 (and in particular I will set  $7 = 7 \bmod 7$  to avoid confusing  $e_0$  with  $e_7$ ). This choice, as it turns out, has an influence on what follows [3,4].

Define

$$E_8^+ = \begin{aligned} & \{\pm e_a\} \\ & \cup \{(\pm e_a \pm e_b \pm e_c \pm e_d)/2 : a, b, c, d \text{ distinct, } e_a(e_b(e_c e_d)) = \pm 1\}, \end{aligned} \quad (11)$$

$$a, b, c, d \in \{0, \dots, 7\}.$$

These  $16 + 14 \times 16 = 240$  elements of the unit octonion 7-sphere form the inner shell of an  $E_8$  lattice, which, like  $D_4^+$  is closed under multiplication [5].

Define

$$E_8^- = \begin{aligned} & \{\frac{1}{2}(\pm e_a \pm e_b) : a, b \text{ distinct}\} \\ & \cup \{\frac{1}{8}(\sum_{a=0}^7 \pm e_a) : \text{odd number of '+'s}\}, \end{aligned} \quad (12)$$

These  $112 + 128 = 240$  elements of the octonion 7-sphere of radius  $1/\sqrt{2}$  also form the inner shell of an  $E_8$  lattice, which, like  $D_4^-$  is not closed under multiplication. (The effect of my choice of octonion multiplication in (10) is in the definition of  $E_8^-$ ; there are choices that would require "odd number of +'s" to be changed to "even number of +'s" in (12); this would not change the order of that set, which would still be 128.)

From these two sets we can construct the inner shell of the 16-dimensional  $\Lambda_{16}$  lattice. In particular,

$$\begin{aligned} & \{\langle U, 0 \rangle, \langle 0, V \rangle : U, V \in E_8^+\} \quad (2 \times 240 = 480 \text{ elements}) \\ \cup & \{\langle U, V \rangle : U, V \in E_8^-, UV^\dagger = \pm e_a\} \quad (16 \times 240 = 3840 \text{ elements}) \end{aligned} \quad (13)$$

( $a \in \{0, \dots, 7\}$ ) is the inner shell of a  $\Lambda_{16}$  lattice, a subset of the unit 15-sphere in  $\mathbf{O}^2$ .

As an example, let  $U = \frac{1}{2}(1 + e_7)$ . Then the 16 values of  $V$  for which  $\langle U, V \rangle \in \Lambda_{16}$  are:

$$\pm U = \pm \frac{1}{2}(1 + e_7), \quad \pm \frac{1}{2}(1 - e_7), \quad \frac{1}{2}(\pm e_1 \pm e_5), \quad \frac{1}{2}(\pm e_2 \pm e_3), \quad \frac{1}{2}(\pm e_4 \pm e_6),$$

the last 14 of which satisfy  $U + V \in E_8^+$ , by which they may also be characterized.

As another example, let  $U = \frac{1}{4}(-1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \in E_8^-$ .

In this case the 16 appropriate  $V$ 's are:

$$\begin{aligned}
\pm U &= \pm \frac{1}{4}(-1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7), \\
&\pm \frac{1}{4}(-1 + e_1 + e_2 - e_3 - e_4 - e_5 + e_6 - e_7), \\
&\pm \frac{1}{4}(-1 - e_1 + e_2 + e_3 - e_4 - e_5 - e_6 + e_7), \\
&\pm \frac{1}{4}(-1 + e_1 - e_2 + e_3 + e_4 - e_5 - e_6 - e_7), \\
&\pm \frac{1}{4}(-1 - e_1 + e_2 - e_3 + e_4 + e_5 - e_6 - e_7), \\
&\pm \frac{1}{4}(-1 - e_1 - e_2 + e_3 - e_4 + e_5 + e_6 - e_7), \\
&\pm \frac{1}{4}(-1 - e_1 - e_2 - e_3 + e_4 - e_5 + e_6 + e_7), \\
&\pm \frac{1}{4}(-1 + e_1 - e_2 - e_3 - e_4 + e_5 - e_6 + e_7).
\end{aligned}$$

Again, the last 14 of these elements may be characterized by  $U + V \in E_8^+$ .

#### 4. More Fibrations.

If  $U, V \in \mathbf{O}$  satisfy

$$UU^\dagger + VV^\dagger = 1,$$

then the doublet  $\begin{bmatrix} U \\ V \end{bmatrix}$  is an element of  $S^{15}$ , the (unit) 15-sphere. As before define the map

$$\begin{aligned}
\begin{bmatrix} U \\ V \end{bmatrix} &\longrightarrow \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}^\dagger = \begin{bmatrix} UU^\dagger & UV^\dagger \\ VU^\dagger & VV^\dagger \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned} \tag{14}$$

$$+ \frac{UU^\dagger - VV^\dagger}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{UV^\dagger + VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{UV^\dagger - VU^\dagger}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The set of all elements

$$\left\langle \frac{UU^\dagger - VV^\dagger}{2}, \frac{UV^\dagger + VU^\dagger}{2}, \frac{UV^\dagger - VU^\dagger}{2} \right\rangle$$

(first two real, third pure octonion, so 9-dimensional) covers  $S^8$  (the 8-sphere in  $\mathbf{R}^9$ , in this case of radius  $\frac{1}{2}$ ). This is the highest dimensional example of a sphere fibration [1]:

$$S^{15} \xrightarrow{S^7} S^8. \tag{15}$$

If  $\begin{bmatrix} U \\ V \end{bmatrix} \in \Lambda_{16}$ , as defined in (13), then the map (14) takes  $\Lambda_{16}$  onto the lattice  $Z^9$

(inner shells again), consisting of elements of the form

$$\begin{aligned} & \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ & \pm \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ or } \pm \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ or } \pm \frac{\epsilon_a}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned} \tag{16}$$

( $a = 1, \dots, 7$ ). This is another example of a lattice fibration:

$$\Lambda_{16} \xrightarrow{E_8} Z^9. \tag{17}$$

## 5. Conclusion.

The reader may be curious to know the purpose of this work. If the reader discovers that purpose before I do, I would ask the reader to let me know. For the nonce, it's just pretty stuff.

## References

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