Tensored Division Algebras: Resolutions of the Identity and Particle Physics

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Invariance groups and multiplets associated with resolutions of the identity of tensored division algebras correspond to groups and multiplets occuring in particle physics.

The real complex division algebras with unity are the complexes **C**, quaternions **H**, and octonions **O**. Being division algebras they contain no divisors of zero (if $x, y \in \mathbf{K}$ a division algebra, and $xy = 0$, then $x = 0$ or $y = 0$). Therefore they can contain no subsets of orthogonal idempotents (projection operators), and there are no nontrivial resolutions of the identity.

Tensor products (over the reals **R**) of pairs of these division algebras are not themselves division algebras, and their identities admit resolutions into sets of nontrivial, orthogonal primitive idempotents. Using these idempotents we can resolve the tensor product algebras themselves into orthogonal subspaces.

Like the division algebras, these tensor product algebras can be viewed as the spinor spaces of Clifford algebras (generated from the algebra of actions of an algebra on itself). The 2-vectors of these Clifford algebras generate the invariance group of an associated pseudo-orthogonal space, and a subgroup of this group will leave invariant the spinor subspaces associated with the resolution of the identity of the spinor space itself (the original tensor product algebra). That is, we have spinor spaces that are also algebras; the algebras admit resolutions of their algebra identities; the projection operators of these resolutions decompose the spinor space into orthogonal subspaces; and our job is to find the invariance groups of some of these decompositions.

For example, let $\mathbf{S} = \mathbf{C} \otimes \mathbf{O}$. The imaginary unit of **C** is denoted i, and the 7 imaginary units of **O** are denoted e_a, $a = 1,...,7$. From i and some unit element in the imaginary subspace of **O** (conventionally chosen to be e_7) we can construct a pair of orthogonal primitive idempotents that resolve the identity of **S**. Define

$$\rho_{\pm} = \frac{1}{2}(1 \pm ie_7). \tag{1}$$

A resolution of the identity into a set of orthogonal projection operators also resolves the algebra itself into orthogonal subspaces. In this case there are four:

$$\mathbf{S}_{++} = \rho_+\mathbf{S}\rho_+; \;\; \mathbf{S}_{+-} = \rho_+\mathbf{S}\rho_-; \;\; \mathbf{S}_{--} = \rho_-\mathbf{S}\rho_-; \;\; \mathbf{S}_{-+} = \rho_-\mathbf{S}\rho_+. \tag{2}$$

These subspaces have an invariance group to be determined below.

C, **H**, **O** and **S** are the spinor spaces of some Clifford algebras. Let $\mathcal{CL}(p,q)$ be the Clifford algebra of of the pseudo-orthogonal space with signature {p(+), q(-)}. For any algebra **K** let $\mathbf{K}_L$ be the algebra of all left actions of **K** on itself, $\mathbf{K}_R$ the algebra of all right actions, and $\mathbf{K}_A$ the algebra of all left, right and combined actions. Let $\mathcal{M}_n(\mathbf{K})$ = n×n matrices over **K**. Then

- $\mathbf{O}_L$, $\mathbf{O}_R$ and $\mathbf{O}_A$ are identical, isomorphic to $\mathcal{M}_8(\mathbf{R}) \simeq \mathcal{CL}(0,6)$; 64-dimensional bases are of the form 1, e_{La}, e_{Lab}, e_{Labc}, or 1, e_{Ra}, e_{Rab}, e_{Rabc}, where, for example, if $x \in \mathbf{O}$, then $e_{Lab}[x] \equiv e_a(e_bx)$, and $e_{Rab}[x] \equiv (xe_a)e_b$ (see **[1]**);
- $\mathbf{H}_L$ and $\mathbf{H}_R$ are distinct, both isomorphic to $\mathbf{H} \simeq \mathcal{CL}(0,2)$;

- $\mathbf{H}_A$ is isomorphic to $\mathcal{M}_4(\mathbf{R}) \simeq \mathcal{CL}(3,1)$;
- $\mathbf{C}_L$, $\mathbf{C}_R$ and $\mathbf{C}_A$ are identical, isomorphic to $\mathbf{C} \simeq \mathcal{CL}(0,1)$ (so we only need use $\mathbf{C}$ itself);
- and $\mathbf{S}_L$, $\mathbf{S}_R$ and $\mathbf{S}_A$ are identical, isomorphic to $\mathcal{M}_8(\mathbf{C}) \simeq \mathcal{CL}(7,0)$.

The elements $\{ie_{La},\ a = 1, ..., 7\}$ are a natural 1-vector basis for $\mathbf{S}_L = \mathbf{S}_A \simeq \mathcal{CL}(7,0)$. In this case the 2-vector basis is the set of 21 distinct $\{e_{Lab}, a \neq b\}$ (note: $a \neq b$ implies $e_{Lab} = -e_{Lba}$, so these elements are not distinct). These elements form a basis for the Lie algebra so(7) and generate the Lie group Spin(7).

The elements of so(7) that leave the $\mathbf{S}_{\pm\pm}$ invariant must commute with the both the $\rho_\pm$, the one acting on $\mathbf{S}$ from the left and the one from the right. Define the idempotents $\rho_{L\pm} = \frac{1}{2}(1 \pm ie_{L7})$ and $\rho_{R\pm} = \frac{1}{2}(1 \pm ie_{R7})$, so $\mathbf{S}_{\pm\pm} = \rho_{L\pm}\rho_{R\pm}[\mathbf{S}]$. If $g \in \mathrm{so}(7)$ leaves each of the $\mathbf{S}_{\pm\pm}$ invariant, then

$$g\rho_{L\pm}\rho_{R\pm} = \rho_{L\pm}\rho_{R\pm}g = \rho_{L\pm}\rho_{R\pm}g\rho_{L\pm}\rho_{R\pm}$$

(the last equality because the ρ's are idempotents). So the subalgebra of so(7) leaving the $\mathbf{S}_{\pm\pm}$ invariant is $\rho_{L\pm}\rho_{R\pm}$ so(7) $\rho_{L\pm}\rho_{R\pm}$, and it was shown elsewhere ([**2**]) that

$$\rho_{L\pm}\rho_{R\pm}\mathrm{so}(7)\rho_{L\pm}\rho_{R\pm} \simeq \{\mathrm{u}(1) \times \mathrm{su}(3)\}\rho_{L\pm}\rho_{R\pm}. \tag{3}$$

With respect to the resulting Lie group SU(3), $\mathbf{S}_{++}$ transforms as a singlet, $\mathbf{S}_{--}$ an antisinglet, $\mathbf{S}_{+-}$ as a triplet, and $\mathbf{S}_{-+}$ as an antitriplet.

Define $\mathbf{D} = \mathbf{C} \otimes \mathbf{H}$. As was true of $\mathbf{S}$, this new combination is not a division algebra and admits a similar resolution of its identity. Let $q_k,\ k = 1, 2, 3$, be a basis for the imaginary subspace of $\mathbf{H}$. Define

$$\lambda_\pm = \frac{1}{2}(1 \pm iq_3), \tag{4}$$

and with these decompose $\mathbf{D}$ into four orthogonal subspaces

$$\mathbf{D}_{++} = \lambda_+\mathbf{D}\lambda_+;\ \ \mathbf{D}_{+-} = \lambda_+\mathbf{D}\lambda_-;\ \ \mathbf{D}_{--} = \lambda_-\mathbf{D}\lambda_-;\ \ \mathbf{D}_{-+} = \lambda_-\mathbf{D}\lambda_+. \tag{5}$$

As was done above for $\mathbf{S}$, we're interested in the subgroup of the group generated by the bivectors of $\mathbf{D}_A \simeq \mathcal{M}_4(\mathbf{C}) \simeq \mathcal{CL}(0,5)$ (note: $\mathbf{D}_A$ is also isomorphic to the Dirac algebra) that leave the subspaces $\mathbf{D}_{\pm\pm}$ invariant. In this case, similar to the case above,

$$\lambda_{L\pm}\lambda_{R\pm}\mathrm{so}(5)\lambda_{L\pm}\lambda_{R\pm} \simeq \{\mathrm{so}(2) \times \mathrm{u}(1)\}\lambda_{L\pm}\lambda_{R\pm} \tag{6}$$

(yes, so(2) $\simeq$ u(1), but below we are going to associate the so(2) with a space-time rotation, and u(1) with a spinor transformation).

Define $\mathbf{T} = \mathbf{C} \otimes \mathbf{H} \otimes \mathbf{O}$. The identity of this algebra admits a more complicated resolution. Let $\vec{x}$ and $\vec{y}$ be arbitrary imaginary unit quaternions, and define $\lambda_0 = \frac{1}{2}(1 + i\vec{x})$, $\lambda_1 = \frac{1}{2}(1 - i\vec{x})$, $\lambda_2 = \frac{1}{2}(1 + i\vec{y})$, and $\lambda_3 = \frac{1}{2}(1 - i\vec{y})$. Define (see [**1**])

$$\Delta_0 = \rho_+\lambda_0,\ \Delta_1 = \rho_+\lambda_1,\ \Delta_2 = \rho_-\lambda_2,\ \Delta_3 = \rho_-\lambda_3. \tag{7}$$

These orthogonal primitive idempotents resolve the identity of $\mathbf{T}$, and with them we can decompose $\mathbf{T}$ into 16 orthogonal subspaces:

$$\mathbf{T}_{mn} = \Delta_m \mathbf{T} \Delta_n,\ m, n = 0, 1, 2, 3 \tag{8}$$

(note: the nonassociativity of $\mathbf{O}$ is not an issue here). $\mathbf{T}$ is the spinor space of $\mathbf{T}_A \simeq \mathcal{M}_{32}(\mathbf{C}) \simeq \mathcal{CL}(11, 0)$. However, rather than find the subalgebra of so(11) that leaves the $\mathbf{T}_{mn}$ invariant, we'll jump ahead to the point of this exercise, which is to consider 2×2 matrices over $\mathbf{T}_A$. Before we do, it is worth pointing out that the form of the Δ_m is thought by myself to be as general as possible, while maintaining certain consistency properties (see [**1**]):

$$(\Delta_m \mathbf{T})\Delta_n = \Delta_m(\mathbf{T}\Delta_n);$$

$$\Delta_m(\Delta_n \mathbf{T}) = (\Delta_m \Delta_n)\mathbf{T}.$$

It is conjectured but not proven that no other inequivalent resolution of the identity of $\mathbf{T}$ exists that satisfies these conditions.

In [**1**] we were interested in the associated algebra of actions $\mathbf{T}_L \simeq \mathcal{M}_{16}(\mathbf{C}) \simeq \mathcal{CL}(0, 9)$, and the 2×2 matrices over this algebra, $\mathcal{M}_2(\mathbf{T}_L) \simeq \mathcal{M}_{32}(\mathbf{C}) \simeq \mathbf{C} \otimes \mathcal{CL}(1, 9)$. Here we'll focus our attention on

$$\mathcal{M}_2(\mathbf{T}_A) \simeq \mathcal{M}_{64}(\mathbf{C}) \simeq \mathcal{CL}(12, 1).$$

Its spinor space is ${}^2\mathbf{T}$, the space of 2×1 column matrices over $\mathbf{T}$. Like $\mathbf{T}$, this can be decomposed into 16 orthogonal subpaces:

$${}^2\mathbf{T}_{mn} = \Delta_m({}^2\mathbf{T})\Delta_n,\ m, n = 0, 1, 2, 3. \tag{9}$$

We'll use an explicit representation of $\mathcal{CL}(12, 1)$. First we need a basis for $\mathcal{M}_2(\mathbf{R})$:

$$\epsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Our basis of 1-vectors for $\mathcal{CL}(12, 1)$ consists of the 13 elements:

$$\gamma,\ \ e_{L7}q_{Lk}\beta,\ \ ie_{Lp}\beta,\ \ iq_{Rj}\alpha,\ k, j = 1, 2, 3,\ \ p = 1, ..., 6. \tag{10}$$

(Note: following what was done in **[1]**, the elements γ, $e_{L7}q_{Lk}\beta$ are identified with normal space-time, and the six $ie_{Lp}\beta$ are extra dimensions carrying su(3) charges. The three $iq_{Rj}\alpha$ are new extra dimensions and will be seen to carry isotope su(2) charges.)

The 2-vectors (the Lie algebra so(12,1)) arising from these 1-vectors are:

$$
\begin{array}{rl}
\text{(Noncompact generators)} & e_{L7}q_{Lk}\alpha,\;\; ie_{Lp}\alpha,\;\; iq_{Rj}\beta, \\
\text{(Compact generators)} & q_{Lk}\epsilon,\;\; e_{Lpq}\epsilon,\;\; q_{Rj}\epsilon, \\
 & ie_{Lp7}q_{Lk}\epsilon,\;\; ie_{L7}q_{Lk}q_{Rj}\gamma,\;\; e_{Lp}q_{Rj}\gamma
\end{array}
\tag{11}
$$

(j,k = 1,2,3, and p,q = 1,...,6).

We want to find the subalgebra of so(12,1) that leaves the ${}^2\mathbf{T}_{mn}$ invariant, but we're going to decompose ${}^2\mathbf{T}$ a little further with the chiral projectors,

$$\eta_\pm = \frac{1}{2}(1 \pm \alpha).$$

Define

$${}^2\mathbf{T}_{mn\pm} = \eta_\pm({}^2\mathbf{T}_{mn}). \tag{12}$$

Ok, let's dig into so(12,1) in steps. We're looking to find the subalgebra

$$
\begin{aligned}
&\eta_\pm \Delta_{Lm}\Delta_{Rn}\text{so(12,1)}\Delta_{Rn}\Delta_{Lm}\eta_\pm \\
&= \lambda_{Lm}\lambda_{Rn}\eta_\pm\rho_{L\pm}\rho_{R\pm}so(12,1)\rho_{R\pm}\rho_{L\pm}\eta_\pm\lambda_{Rn}\lambda_{Lm},
\end{aligned}
\tag{13}
$$

which maps the subspace ${}^2\mathbf{T}_{mn\pm}$ to itself (note: the $\pm$'s on one side of so(12,1) are independent, but linked to the corresponding $\pm$'s on the opposite side of so(12,1)). We'll do this in steps, starting with $\eta_\pm$so(12,1)$\eta_\pm$. Since $\eta_\pm(\gamma \text{ or } \beta)\eta_\pm = \eta_\pm\eta_\mp(\gamma \text{ or } \beta) = 0$, we are left with generators,

$$
\begin{array}{rl}
\text{(Noncompact generators)} & e_{L7}q_{Lk}\alpha,\;\; ie_{Lp}\alpha, \\
\text{(Compact generators)} & q_{Lk}\epsilon,\;\; e_{Lpq}\epsilon,\;\; q_{Rj}\epsilon, \\
 & ie_{Lp7}q_{Lk}\epsilon.
\end{array}
\tag{14}
$$

Since the generators $q_{Rj}\epsilon$ commute with the other remaining generators, we're left with

$$\eta_\pm\text{so(12,1)}\eta_\pm = (\text{so(9,1)} \times \text{su(2)})\eta_\pm. \tag{15}$$

Our penultimate step is:

$$
\begin{aligned}
&\rho_{L\pm}\rho_{R\pm}[(\text{so(9,1)} \times \text{su(2)})\eta_\pm]\rho_{L\pm}\rho_{R\pm} \\
&= (\text{so(3,1)} \times \text{u(1)} \times \text{su(2)} \times \text{su(3)})\eta_\pm\rho_{L\pm}\rho_{R\pm}
\end{aligned}
\tag{16}
$$

(note: $\rho_{L\pm}(e_{Lp} \text{ or } e_{Lp7})\rho_{L\pm} = \rho_{L\pm}\rho_{L\mp}(...) = 0$, so the generators $ie_{Lp}\alpha$ and $ie_{Lp7}q_{Lk}\epsilon$ are killed by this reduction, leaving us with so(3,1) $\times$ so(6) {from the $e_{Lpq}\epsilon$} $\times$ su(2) {from the $q_{Rj}\epsilon$}, and $\rho_{R\pm}$ so(6) $\rho_{R\pm}$ reduces to u(1) $\times$ su(3) (see **[2]** and above). Obviously at this point, if one has not done so already, one might be thinking that here we have a bizarre coincidence, since this looks a lot like groups that play a dominant role in the real world. Well well, how bizarre, how bizarre.

Nesting this result inside $\lambda_{Lm}\lambda_{Rn}...\lambda_{Lm}\lambda_{Rn}$ reduces so(3,1) to a boost, and a space rotation about the axis of the boost, and it reduces su(2) to u(1), leaving us with u(1) $\times$ u(1) $\times$ su(3) as the final internal symmetry, which is the exact part of the standard symmetry.

As a mathematical curiosity **[3]** we'll use this machinery to connect to the exceptional group E_6. We start by noting that

$$\mathcal{M}_2(\mathbf{T}_A) \simeq \mathcal{M}_{64}(\mathbf{C}) \simeq \mathcal{CL}(0,13).$$

Below is a 1-vector basis for this $\mathcal{CL}(0,13)$:

$$\gamma,\;\; ie_{L7}q_{Lk}\beta,\;\; e_{Lp}\beta,\;\; q_{Rj}\alpha,\; k,j=1,2,3,\;\; p=1,...,6. \tag{17}$$

It is an obvious modification of the 1-vector basis of $\mathcal{CL}(12,1)$, and its corresponding 2-vector (so(13)) basis is:

$$\begin{aligned} &ie_{L7}q_{Lk}\alpha,\;\; e_{Lp}\alpha,\;\; q_{Rj}\beta,\\ &q_{Lk}\epsilon,\;\; e_{Lpq}\epsilon,\;\; q_{Rj}\epsilon,\\ &ie_{Lp7}q_{Lk}\epsilon,\;\; ie_{L7}q_{Lk}q_{Rj}\gamma,\;\; e_{Lp}q_{Rj}\gamma \end{aligned} \tag{18}$$

(j,k = 1,2,3, and p,q = 1,...,6).

As we did above, we'll use projection operators to simultaneously reduce the 128-dimensional spinor space of this Clifford algebra ($^2\mathbf{T}$), and to find the subalgebra of so(13) leaving the reduced spinor subspaces invariant - but we won't go quite as far. The first reduction is $(^2\mathbf{T}) \longrightarrow \eta_\pm(^2\mathbf{T})$, which breaks $(^2\mathbf{T})$ into two 64-dimensional subspaces. The corresponding reduction on so(13) is $\eta_\pm$so(13)$\eta_\pm$. The surviving 2-vectors are

$$\begin{aligned} &ie_{L7}q_{Lk}\alpha,\;\; e_{Lp}\alpha,\\ &q_{Lk}\epsilon,\;\; e_{Lpq}\epsilon,\;\; q_{Rj}\epsilon,\\ &ie_{Lp7}q_{Lk}\epsilon \end{aligned} \tag{19}$$

(each with an $\eta_\pm$). This is a basis for the Lie algebra (so(10)$\times$su(2))$\eta_\pm$, where as before the three elements $q_{Rj}\epsilon$ are a basis for su(2).

Each of the two 64-dimensional spinor spaces $\eta_\pm(^2\mathbf{T})$ is an su(2) doublet of 32-dimensional so(10) spinors. It is widely known that the Lie algebra of the exceptional group E_6 can be formed in a natural way from so(10), a 32-dimensional spinor space, and a copy of u(1). Using the projection operators $\lambda_\pm = \frac{1}{2}(1 \pm iq_3)$ and their right-action counterparts, $\lambda_{R\pm} = \frac{1}{2}(1 \pm iq_{R3})$, we can project a pair of 32-dimensional so(10) spinors $\eta_\pm(^2\mathbf{T})\lambda_\pm = \lambda_{R\pm}\eta_\pm[^2\mathbf{T}]$ from the su(2) doublet $\eta_\pm(^2\mathbf{T})$. This is associated with the Lie algebra reductions:

$$\lambda_{R\pm}\eta_\pm\mathrm{so}(13)\eta_\pm\lambda_{R\pm} = \lambda_{R\pm}(\mathrm{so}(10)\times\mathrm{su}(2))\eta_\pm\lambda_{R\pm} = (\mathrm{so}(10)\times\mathrm{u}(1)))\eta_\pm\lambda_{R\pm}. \tag{20}$$

In each of four reductions we are left with a Lie algebra so(10)×u(1) acting on a 32-dimensional spinor space, and from each of these collections we can construct a Lie algebra for E_6.

Although all generators are compact in this case, were E_6 of this form required to make a reasonable physics model, it is interesting to note that it's presence is associated with the breaking of su(2) down to u(1).

References:

[1] G.M. Dixon, *Division Algebras: Octonions, Quaternions, Complex Numbers, and the Algebraic Design of Physics*, (Kluwer, 1994).

[2] G.M. Dixon, www.7stones.com/Homepage/10Dnew.pdf

[3] Inspired by private communications with John Baez and Tony Smith.